Field Theories On Curved Spacetimes With Boundaries

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FERNANDES KARAN ARTHUR

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Contents

1 Introduction and Overview

Quantum field theory and the General theory of Relativity provide us with our best descriptions of the four fundamental forces in Nature. With the exception of gravity, the Standard Model of particle physics has successfully quantized all other known forces, namely the strong, weak and electromagnetic forces. Gauge fields are of particular significance within quantum field theory, since it is their quanta – the gauge bosons, which mediate interactions described by the Standard Model. On the other hand, the General theory of Relativity describes gravity as curvature, resulting from a classical, dynamical spacetime metric. The solutions of Einstein's equations involve curved backgrounds which have globally defined null surfaces, such as black hole backgrounds and our observable universe.

Spectral observations of Type Ia supernovae [1, 2] indicate that our universe is expanding at an accelerated rate. This observation can be accounted for by the Einstein field equations which include a positive cosmological constant Λ . An accelerating universe further implies that our observations are confined to be within a cosmological horizon. We are also aware that our universe is populated by black holes. Several candidate binary black hole systems have been detected indirectly using X-ray astronomy over the years [3–5]. More recently, the Laser Interferometer Gravitational-Wave Observatory (LIGO) have detected the transient gravitational-wave signals corresponding to the merger of inspiralling binary black hole systems [6–

10]. The gravitational wave observations of these events are completely consistent with the properties of black holes predicted by General Relativity [11].

The outer null surface of a black hole is an example of a Killing horizon, called the event horizon, which prevents classical observations of its interior from being made by stationary observers outside the black hole. The cosmological horizon is another example of a Killing horizon, which confines our observations to within a cosmological neighbourhood of our entire universe. Thus Killing horizons are globally defined null surfaces of the spacetime which are an effective boundary to our observations. While the implications of spatial boundaries on field theories have been investigated over the years, the effect of Killing horizons on field theories, and in particular gauge theories, have not been well understood.

Spatial boundaries are relevant in the dynamics and quantization of field theories, especially in the case of gauge fields. For example in classical electrostatics, the potential in the presence of conducting surfaces is provided by the solution of the Laplace equation in a space with boundaries. Equivalently, we can state that the total charge in a region is given by the electric flux across the boundary enclosing all the charges. Another example involves the Chern-Simons theory on a disk, where the boundary modifies the classical dynamics and vacuum structure of the quantum theory [12,13]. In the above examples concerning the Maxwell field, the dynamics of the fields are subject to the Gauss law constraint of the theory. While the Gauss law constraint is not affected by the presence of spatial boundaries, it implies that the fields must satisfy certain boundary conditions to be consistent with the constraint. As a consequence, this affects the gauge transformations and physical states of the Maxwell field at the boundary.

The situation in the case of horizons is quite different. Killing horizons are causal boundaries for stationary or static observers outside the event horizon of a black hole, or within a cosmological horizon. However, they are not physical boundaries as in the case of spatial boundaries. In particular, the spacetime manifold exists well beyond the Killing horizons. Let us consider for instance a freely falling observer into a black hole. Classical General Relativity tells us that this observer finds nothing special at the event horizon of a black hole, where curvature invariants of the background are in fact well behaved. Thus components of the stress tensor, or more specifically scalar invariants such as $T_{\mu\nu}T^{\mu\nu}$ are also well behaved at the horizons. In the case of matter field theories, we can make specific assumptions on the behaviour of the fields at the horizon. On the other hand, we cannot a priori impose such conditions on constrained field theories. Only gauge invariant scalars constructed out of the fields remain finite at the horizon, while the gauge fields themselves may remain completely arbitrary. Boundary conditions on gauge fields can only be imposed after deriving the constraints. In particular, the gauge parameters or their derivatives need not vanish at the horizons of the background. This property allows for the modification of the constraints of field theories, such as the Gauss law constraint, at the Killing horizons of the spacetime. The central aim of this thesis is to explore these modifications and their effect on the dynamics and quantization of gauge theories on backgrounds with one or more Killing horizons.

Dirac and independently Bergmann and collaborators developed the Hamiltonian formulation for constrained systems [14–16], which was in particular used to investigate the dynamics and possible quantization of the gravitational field [17–21]. Building on these works, Arnowitt, Deser and Misner developed the $3+1$ formalism in terms of the shift and lapse variables [22]. This formalism was used on the gravitational action to define the ADM mass, momentum and angular momentum as surface integrals evaluated at spatial infinity. Regge and Teitelboim further demonstrated that these surface integrals are necessary for a consistent Hamiltonian formulation of General Relativity on asymptotically flat backgrounds. More specifically, the surface integrals are a consequence of the constraints of General Relativity [23, 24]. The Dirac-Bergmann formalism has been considered for field theories on curved backgrounds with spatial boundaries [12, 25–28]. As in the case of constrained theories on flat backgrounds, the constraints of gauge and gravitational fields are not modified in the case of spatial boundaries. Any surface terms which arise in the Hamiltonian formalism are identified with additional boundary conditions to be imposed on the fields [27, 28]. As will be argued in this thesis, Killing horizons differ from spatial boundaries in that they modify the constraints of field theories. Consequently, known results on black hole backgrounds could also be modified, especially at the horizons.

Based on the conserved charges of gauge and gravitational fields on asymptotically flat backgrounds, Wheeler conjectured that black hole horizons have no (classical) hair [29]. This conjecture asserts that black holes can be only characterized by their mass, charge and angular momentum, all of which are conserved quantities associated with a Gauss law constraint. The absence of hairs on the horizons of black holes implies that any internal configuration of a black hole, that which lies behind the event horizon, cannot be determined from external observations. However, many properties of black holes have been understood from the interaction of their horizons with external fields and perturbations.

For instance, it is known that black holes in the presence of external electromagnetic fields have an induced surface charge and current [30, 31]. The horizons of black holes also have certain mechanical properties. External gravitational fields can tidally deform rotating black holes, which leads to the conclusion that a horizon can also behave like a viscous fluid [32,33]. Black holes also have a mass and surface area which always increases in any dynamical process outside the horizon [34–36].

These results concerning the mass and area led Bekenstein to suggest that black holes possess an entropy proportional to the surface area of the event horizon [37]. In considering quantum fields outside the event horizon of a stationary black hole, Hawking further discovered that black holes have a temperature and will radiate a thermal distribution of field quanta [38]. This result is a part of the correspondence between the laws of black hole mechanics and thermodynamics, through which black holes can be shown to satisfy all four laws of thermodynamics [39]. Damour successfully rewrote the equations governing the evolution of general black hole horizons in the form of electromagnetic, mechanical and thermodynamic equations [40].

The above relations satisfied at the horizons of black holes were incorporated into the $3+1$ formulation of General Relativity through the membrane paradigm [41]. The dynamics of matter fields in the standard $3+1$ formalism on black hole backgrounds are often 'frozen' at the horizon. In addition, quantum fields around black holes can suffer from divergences at the horizon [42]. In the membrane paradigm, the black hole interior and horizon are replaced with a timelike physical membrane, endowed with the known electrical, mechanical and thermodynamical properties of horizons [43]. The properties of fields at the membrane affect observations made by stationary observers outside the black hole within the membrane paradigm. As an example, the driving of an accretion disk into a Kerr black holes equatorial plane has been understood as a consequence of both the black hole's 'gravitomagnetic' field outside the membrane and the disk's viscosity at the membrane [44]. The membrane paradigm has also helped provide insights into the entropy and thermal atmosphere surrounding black holes as they evaporate through Hawking radiation [45, 46].

The membrane paradigm approximates the black hole horizon as a timelike surface. As stated previously, my thesis will consider how the constraints of field theories are modified by the presence of Killing horizons. Unlike spatial boundaries, we will find that Killing horizons provide additional contributions to the constraints of field theories. Thus constraints from the horizon will have implications on the conserved charges and the properties of fields at the horizons, which will differ from the predictions for gauge fields at the membrane following the membrane paradigm. Horizon corrections to the constraints could also be relevant in the quantum description of black holes. We recall that the radiation emitted by black holes can be described by the renormalized expectation value of the stress-energy tensor $\langle \hat{T}_{\mu\nu} \rangle_{\mathrm{ren}}$ near the horizon [46–49]. The expectation value is determined with respect to the vacuum state of the curved background. However, unlike Minkowski spacetime, vacuum states on curved backgrounds are not globally defined and modes are required to satisfy appropriate boundary conditions at the horizon and null infinity [50–52]. While such conditions can be unambiguously applied on the modes of matter field theories, care must be taken in the case of gauge theories. Contributions from the horizons in the constraints have a direct implication on the gauge invariant quantum state of the theory at the horizons. Furthermore, since boundary conditions on gauge fields must respect the constraints at the horizons, they are not as freely specifiable as in the case of matter fields.

Fields at the horizon might also be related to the entropy of black holes. For instance, Carlip considered the constraints of General Relativity and demonstrated that the surface algebra of diffeomorphisms at the horizons of $2 + 1$ dimensional black holes contains a Virasaro subalgebra. The central charge of the surface algebra was shown to be proportional to the Bekenstein-Hawking entropy [53, 54]. An understanding of the microstates which account for the entropy of black holes might provide a resolution to the longstanding information paradox, which is based on Hawking's observation that should a black hole completely radiate away, it would imply a loss of information and unitarity [55]. While there have been several attempts at an explanation over the decades, many of which make use of properties of generic fields at the horizon [56–61], the paradox has remained unresolved. A recently proposed resolution involves soft hairs on the horizons of black holes [62, 63]. This proposal is based on relations between the symmetries at null infinity and Weinberg's soft theorems for gauge and gravitational fields on asymptotically flat backgrounds [64–66]. Specifically, the soft theorems for photons and gravitons were shown to result as the Ward identities associated with infinite-dimensional symmetry groups at null infinity $\mathscr{I}$. These involve large gauge transformations that approach angle dependent constants at $\mathscr{I}$ for abelian gauge theories and the supertranslations of the BMS group for gravitational theories. The corresponding conserved currents imply the existence of soft charges on the sphere at null infinity. It has been argued that similar soft charges might exist on the horizons of black holes, implying the presence of soft hairs which could retain information [67].

A possible description of soft charges may be realized through dressed scalar and fermionic fields. Faddeev and Kulish extended previous results in the literature [68, 69] to construct fermionic fields dressed with soft photons which provide a non-vanishing S-matrix [70]. This is in contrast with the Fock basis S-matrix elements of quantum electrodynamics, which are known to vanish [71, 72]. The dressed fields involved in the Faddeev-Kulish construction have been shown to be consistent with Weinberg's soft photon theorem and provide a realization of the soft charges at null infinity [73]. However, the construction of dressed charges which could describe soft hairs on the horizons of black holes remains an open problem. The Faddeev-Kulish dressing terms have also been derived for static charges of quantum electrodynamics in flat spacetime by using the radiation gauge [74, 75] and within the BRST formalism [76–78]. This suggests that soft charges and their physical implications could be further investigated using well developed formalisms for gauge theories. In this thesis, we will derive the Gauss law constraint of the Maxwell field which involves surface contributions from the horizons of the background. A good gauge fixing choice in this case involves the radiation gauge with additional surface terms from the horizons. We will demonstrate that this gauge modifies the known dressing function of the fields in scalar quantum electrodynamics on flat backgrounds.

In light of current observations, which were mentioned at the beginning of this chapter, it is of interest to generalize known results on asymptotically flat black hole backgrounds to black hole backgrounds with a cosmological horizon. We can always consider a $3+1$ decomposition (also known as a foliation) of spacetime into spacelike hypersurfaces and 'time'. On backgrounds with certain symmetries, such as spherically symmetric and axisymmetric backgrounds, we can further consider foliations where spatial sections of the horizons of the background are the boundaries of the spacelike hypersurfaces. One proven way to investigate the dynamics and charges of gauge theories on foliated backgrounds is through the constrained Hamiltonian formulation of field theories. The significant difference in the treatment of constrained field theories on backgrounds with spatial boundaries and those which will be considered in my thesis involves the nature of the boundary.

While boundary conditions ensure the regularity of the fields at the boundary of a manifold, they are restrictive for gauge fields in general. Any value ascribed to gauge fields can be altered by gauge transformations. Boundary conditions can be chosen as a gauge fixing choice. However, such conditions should be chosen only after a determination of the constraints of the theory. The Dirac-Bergmann formalism on field theories also requires the evaluation of Poisson brackets involving smeared constraints. The smearing functions are in the same space as the parameters of gauge transformations and in the dual space of the constraints which generate gauge transformations. With spatial boundaries, either the smearing function or their derivatives vanish at the boundary to ensure the regularity of the fields there. Killing horizons on the other hand are globally defined null surfaces located within a given manifold. The smearing functions and gauge fields at the horizons can be completely arbitrary, so long as gauge invariant scalars constructed from the fields are finite. As a consequence of smearing functions which do not vanish at the horizons, we will demonstrate that the constraints of gauge theories can involve surface modifications due to the horizons of curved backgrounds. Such modifications will further affect the dynamics and observed charges of constrained field theories.

We will review the Dirac-Bergmann formalism for constrained field theories on general curved backgrounds in the next chapter, following the classic treatment provided in [16, 79–82]. Our review will set up the covariant notation and framework which will be adopted in subsequent chapters of the thesis. The formalism as described in this chapter will be applicable to any foliated curved background involving spatial hypersurfaces on which the Hamiltonian is defined. The review will also consider Grassmannian fields, which allows for the treatment of both bosonic or fermionic variables. This will be particularly relevant in the final chapter of this thesis on the Hamiltonian BRST formalism, in which the BRST charge and the ghosts have odd Grassmann parity .

In Chapter 3, we will consider the Dirac-Bergmann formalism on spherically symmetric spacetimes with one or more horizons [83]. The foliation is carried out with respect to the timelike Killing vector field of the spacetime, whose norm vanishes on the horizons of the background. This leads to spatial hypersurfaces whose boundaries correspond to spatial sections of the horizons of the spacetime. As examples, we will consider the Maxwell and Abelian Higgs fields. In the case of both field theories, we demonstrate that the Gauss law constraint involves additional surface contributions from the horizons of the spacetime. The surface terms in the constraints lead to gauge transformations of the fields which retain their usual form. These surface contributions due to the horizons however do affect the observed charge. The gauge fixing of the theory can also be chosen to include surface terms at the horizons. We will see that in some cases, such surface terms are necessary to ensure that the gauge has been appropriately fixed at the horizons.

The integration of the Gauss law constraint provides an expression for the charge. By integrating the Gauss law constraint of the Maxwell field over a volume outside the black hole horizon and within the cosmological horizon, we find the usual expression for the electric flux across the outer boundary of the volume, which is a closed spatial surface. However, we also demonstrate that the surface term in the constraint causes the electric flux to vanish across the horizons. This suggests the interpretation that equal and opposite charges are present on either side of the horizon, with the charges behind the horizon screened for an external observer. In the case of the Abelian Higgs field, we find a similar result – a non-vanishing electric flux outside the horizon and a vanishing flux across the horizon. This result holds for the Higgs field in the false vaccuum on black hole de Sitter backgrounds, where an electric flux is present in the background.

We derive the Dirac brackets in the radiation gauge for the Maxwell field and the unitary gauge for the Abelian Higgs field. These brackets are the covariant generalizations of those known in flat spacetime [80]. The modified Gauss law constraint however allows us to consider more general gauge fixing conditions. We consider this possibility in the case of the Maxwell field. The radiation gauge is modified to include an additional surface term analogous to that present in the Gauss law. In this case, the Dirac bracket involves the Green function of the spatial Laplacian of the hypersurface. We consider the limit of the radiation gauge Dirac brackets about the Schwarzschild background when any one of its arguments is evaluated at the horizon. While the usual radiation gauge Dirac bracket reduces to the Poisson bracket in this limit, the modified radiation gauge Dirac bracket involves an additional non-vanishing contribution from the horizon [84].

In Chapter 4, we will consider the Dirac-Bergmann formulation on Kerr backgrounds with one or more horizons [85]. The foliation is carried out with respect to a timelike combination of the temporal and axial Killing vector fields of the background. While this vector coincides with the Killing vector field of the background at the horizons, it is not Killing for all other points on the spacetime. This leads to certain subtleties involved in the Hamiltonian formulation on Kerr backgrounds in comparison with the spherically symmetric case, which is discussed at length. The Gauss law constraint of the Maxwell field involves additional surface contributions from the horizons, similar to the case of spherically symmetric backgrounds. We derive the Dirac brackets of the theory in the axial gauge. These brackets involve specific functions, whose solutions on the asymptotically flat Kerr background is provided in the Appendix.

The results of Chapter 3 motivated us to consider the Hamiltonian BRST formalism on spherically symmetric backgrounds with horizons in Chapter 5 [86]. The BRST formalism in particular can be used to investigate the physical observables and quantization of the theory. We first describe the formalism and the extended phase space involved for all theories of the Yang-Mills type, i.e. theories whose first-class constraints satisfy a Lie algebra. The conventions we adopt provide the usual covariant action in the absence of any horizon contributions in the constraints and gauge fixing terms. We then consider the specific examples of the Yang-Mills field and scalar electrodynamics. By using the Dirac-Bergmann formalism, we find that the Gauss law constraints in these theories also involve surface contributions from the horizons of the background. We then introduce the ghosts and their momenta in the extended phase space and define the BRST charge. The inclusion of surface terms at the horizons in the gauge fixing function is shown to provide effective ghost and gauge fixing actions with surface integrals at the horizons.

We investigate the renormalizability in the case of the Yang-Mills field. Specifically, we are interested in the possible effect of the additional surface integrals at the horizons on the renormalizability of the theory. We first use the Zinn-Justin equation to demonstrate that the renormalized BRST transformations take the same form as the known BRST transformations of the Yang-Mills field. We then consider the invariance of an effective action under the renormalized BRST transformations. The bulk contribution to the effective action of the Yang-Mills field is assumed to have a similar form as that derived using the Hamiltonian BRST formalism. The effective action also includes all possible surface integrals which involve the ghosts and the conjugate momentum of the Lagrange multiplier. We find that an effective action which includes these surface integrals at the horizons is renormalizable.

In the case of scalar electrodynamics, we consider the effect of surface contributions on the physical charges of the theory. Within the Hamiltonian BRST formalism, we can construct a co-BRST charge – a gauge fixing fermion which is also a nilpotent operator. We use the co-BRST charge to identify the dressed scalar fields of the theory which are both BRST and co-BRST invariant. When surface terms are present in the gauge fixing function, we demonstrate that the dressing involves additional surface contributions from the horizons of the background. This provides a generalization of the dressing function of static charges in flat spacetime [74, 75] to spherically symmetric backgrounds which involve one or more horizons.

We summarize the results of the previous chapters and some future directions worth exploring in the conclusion of the thesis.

2 The Dirac-Bergmann formalism on foliated backgrounds

In the case of certain dynamical systems, it may not be possible to solve for all the velocities of the theory in terms of their canonically conjugate momenta. In such theories, the Legendre transform of the Lagrangian to the Hamiltonian is singular and the dynamics of the theory is said to be constrained. The canonical formulation required to treat constrained theories was originally developed by Dirac [14, 16] and independently by Bergmann and collaborators [15] in flat spacetime. The Dirac-Bergmann formalism allows for a systematic derivation of all the constraints of the theory. As the constraints involve relations among the phase space variables, this leads to a Hamiltonian formulation of the theory on a reduced subspace in phase space called the constrained subspace. The constraints of the theory may be either first-class or second-class. Within the canonical framework, gauge transformations are generated by the first-class constraints of the theory. The canonical Hamiltonian evolves a given initial configuration of the fields into a class of final configurations, all of which are related to one another by gauge transformations. To ensure that the dynamical evolution is uniquely defined, we can either 'solve' the first class constraints, or more systematically, introduce additional constraints in the theory to 'fix' a gauge.

Unlike first-class constraints, second-class constraints are not associated with any gauge transformations of the theory. While the Poisson brackets of first-class constraints define an algebra in phase space, the Poisson brackets of second-class constraints do not and need to be consistently eliminated from the theory. This is achieved through the definition of a modified Poisson bracket, called the Dirac bracket, which vanishes when any one of its arguments is a second-class constraint. The Dirac brackets define a new algebra in phase space and are particularly important in the context of gauge fixed theories. Gauge fixing introduces additional constraints which have non-vanishing Poisson brackets with the existing first-class constraints of the theory. The resulting constraints are all second-class and it is the Dirac brackets which define the dynamics of the gauge fixed theory.

In the following sections, I will review the Dirac-Bergmann formalism for field theories following many of the classic references on the subject [79–82]. The treatment will be considered on foliated backgrounds, which will help set up the conventions for subsequent chapters of my thesis. Beginning with the next section, the Hamiltonian formulation for theories which involve Grassmannian fields will be considered on spatial hypersurfaces resulting from a foliation of general curved backgrounds. I will then provide the covariant definitions for the canonical Hamiltonian and Poisson brackets on these spatial hypersurfaces. The review of the Dirac-Bergmann formalism for constrained field theories will then be provided using the definitions introduced for foliated backgrounds. The review will consider topics relevant to this thesis, which include the derivation of the constraints, properties of first-class and second-class constraints and the construction of Dirac brackets. Further details of constrained field theories can be found in the above listed references on the subject.

2.1 Hamiltonian formulation for general field theories

Let us consider the action functional for N fields Φ_A $(A = 1, \cdots, N)$ and their derivatives defined on a Lorentzian spacetime manifold $\mathcal{M}$

$$S[\Phi_A] = \int dV_4^x \; \tilde{\mathcal{L}}(\Phi_A(x)) \,, \tag{2.1}$$

where dV_4 and $\tilde{\mathcal{L}}$ refer to the covariant volume element and Lagrangian density, respectively. We assume that our spacetime manifold can be expressed as $\mathcal{M} = \Sigma \times \mathbb{R}$, with 'time' along $\mathbb{R}$ and Σ as constant time spatial hypersurfaces. This foliation will be particularly relevant in later chapters where we will consider spherically symmetric and axially symmetric backgrounds, both of which involve Killing vector fields that admit such a foliation. The spacetime metric g_{ab} can be written in terms of a spatial metric h_{ab} of the hypersurface Σ and a unit timelike normal u_a to the hypersurfaces, such that

$$g_{ab} = h_{ab} - u_a u_b \,, \tag{2.2}$$

where $u^a u_a = -1$. We can also define the projection operator h_a^b as

$$h_a^b = \delta_a^b + u^b u_a \,, \tag{2.3}$$

where δ_a^b is the Kronecker delta function. Any spacetime tensor can be projected onto the hypersurface Σ using the projection operator given in Eq. (2.3). Time derivatives of the fields $\dot{\Phi}_A$ are defined as the Lie derivative with respect to some time evolution vector t^a of the background. This vector can have both spatial and temporal components, i.e. $t^a = N u^a + N^a$, where $N = -t^a u_a$ is called the lapse and $N^a = t^b h_b^a$ is called the shift of the time evolution vector t^a [87]. Using dV_x to denote the covariant volume element of the hypersurface Σ, we can express the action in Eq. (2.1) in the following way

$$S[\Phi_A] = \int dt\, L[\Phi_A\,, \dot{\Phi}_A] = \int dt \int\limits_{\Sigma} dV_x\; \mathcal{L}(\Phi_A(x)\,, \dot{\Phi}_A)\,, \tag{2.4}$$

where L and $\mathcal{L}$ respectively denote the Lagrangian and the Lagrangian density following the foliation and are related by

$$L[\Phi_A\,, \dot{\Phi}_A] = \int\limits_{\Sigma} dV_x\, \mathcal{L}(\Phi_A(x)\,, \dot{\Phi}_A(x))\,. \tag{2.5}$$

We note that $dV_4^x = N dt dV_x$, where N as before refers to the lapse function. Thus the projection of the action actually produces $N\widetilde{\mathcal{L}}$ for some Lagrangian density $\widetilde{L}$. In Eq. (2.5), we have denoted $N\widetilde{\mathcal{L}}$ as $\mathcal{L}$ in the definition of the Lagrangian.

Subsequent chapters of this thesis will concern foliated backgrounds whose spacelike hypersurfaces Σ have a boundary $\partial\Sigma$ corresponding to spatial sections of the horizons of the background. This cannot be defined generally and requires the consideration of spacetime backgrounds which possess certain symmetries and in particular timelike Killing vector fields. In this thesis, spherically symmetric backgrounds will be considered in Chapters 3 and 5, while a certain class of axisymmetric backgrounds will be considered in Chapter 4, where such foliations can be defined.

We will also let the fields be either bosonic or fermionic. Thus the fields in general belong to a Grassmann algebra. These fields can be assigned either an even or odd Grassmann parity. Denoting the parity by ϵ, we say that the field Φ_A is even when $\epsilon_{\Phi_A} = 0$ (mod 2) and that it is odd when $\epsilon_{\Phi_A} = 1$ (mod 2). Lagrangians and Hamiltonians will always be an even functional of the fields. Because parity is additive for composite fields, given any two functionals of the fields $F(\Phi_A)$ and $G(\Phi_A)$, we have

$$FG = (-1)^{\epsilon_F \epsilon_G} GF\,. \tag{2.6}$$

Due to the presence of odd Grassmanian fields, the variations and derivatives have to be handled carefuly. The variation of a functional $F(\Phi_A)$ of a field Φ_A can be written in two possible ways

$$\text{either} \quad \frac{\delta_L F}{\delta \Phi_A}\,, \quad \text{or} \quad \frac{\delta_R F}{\delta \Phi_A}\,, \tag{2.7}$$

where $\dfrac{\delta_L}{\delta \Phi_A}$ and $\dfrac{\delta_R}{\delta \Phi_A}$ denote the left functional derivative and right functional derivative with respect to Φ_A, respectively. The left functional derivative $\dfrac{\delta_L}{\delta \Phi_A}$ simply entails that we vary F in Eq. (2.7) with respect to Φ_A, with $\delta\Phi_A$ moved to the extreme left and then deleted. Likewise $\dfrac{\delta_R}{\delta \Phi_A}$ in Eq. (2.7) means that we vary F with respect to Φ_A, with $\delta\Phi_A$ moved to the extreme right and then deleted. These variations are identical when the field Φ_A is even. In the following, functional variations and derivatives which are left unspecified will always be taken to mean 'left'. With these definitions at hand, we can now consider the momenta Π^A canonically conjugate to the fields Φ_A. These are defined by

$$\Pi^A = \frac{\delta L}{\delta \dot{\Phi}_A}\,, \tag{2.8}$$

where the functional derivative in this definition is taken on the hypersurface Σ, i.e. it is an 'equal-time' functional derivative

$$\frac{\delta \Phi_A(\vec{x},t)}{\delta \Phi_B(\vec{y},t)} = \delta_A^B\, \delta(x,y) = \frac{\delta \dot{\Phi}_A(\vec{x},t)}{\delta \dot{\Phi}_B(\vec{y},t)}\,. \tag{2.9}$$

We will sometimes refer to the spacetime coordinates x as $x = (\vec{x}, t)$, as in Eq. (2.9). The $\delta(x,y)$ in Eq. (2.9) is the covariant three-dimensional delta function on Σ which satisfies

$$\int\limits_{\Sigma} dV_y\ \delta(x,y) f(\vec{y},t) = f(\vec{x},t)\,, \tag{2.10}$$

for any well behaved function f defined on Σ. Using the Lagrangian L of Eq. (2.5), we can construct the canonical Hamiltonian through the Legendre transform

$$H_C = \int\limits_{\Sigma} dV_x\ (\Pi^A \dot{\Phi}_A) - L\,. \tag{2.11}$$

The repeated indices in Eq. (2.11) and throughout the thesis will imply the sum over all values of the index, in accordance with the summation convention. The canonical Hamiltonian is defined in a $2N$ infinite-dimensional phase space. To describe the dynamics of fields in phase space, we require the definition of Poisson brackets.

The graded Poisson bracket for two functionals of the canonical variables $F\left(\Phi_A, \Pi^A\right)$ and $G\left(\Phi_A, \Pi^A\right)$ will be defined as

$$[F, G]_P = \int_\Sigma dV_z \left(\frac{\delta_R F}{\delta \Phi_A(z)} \frac{\delta_L G}{\delta \Pi^A(z)} - \frac{\delta_R F}{\delta \Pi^A(z)} \frac{\delta_L G}{\delta \Phi_A(z)} \right) . \tag{2.12}$$

Given three functionals of the canonical variables, $F\left(\Phi_A, \Pi^A\right)$, $G\left(\Phi_A, \Pi^A\right)$ and $H\left(\Phi_A, \Pi^A\right)$, the graded Poisson bracket satisfy the following useful relations

$$\begin{aligned}
[F, G]_P &= (-1)^{\epsilon_F \epsilon_G + 1} [G, F]_P \ , \\
[F, GH]_P &= [F, G]_P H + (-1)^{\epsilon_F \epsilon_G} G [F, H]_P \ , \\
[[F, G]_P, H]_P + (-1)^{\epsilon_F(\epsilon_G + \epsilon_H)} [[G, H]_P, F]_P &+ (-1)^{\epsilon_H(\epsilon_F + \epsilon_G)} [[H, F]_P, G]_P = 0 \, , \\
\epsilon\left([F, G]_P\right) &= \epsilon_F + \epsilon_G \, .
\end{aligned} \tag{2.13}$$

The first and last equations of Eq. (2.13) tells us that the graded Poisson bracket represents the commutator or anticommutator in the case of even or odd Grassmannian fields, respectively. The second equality of Eq. (2.13) refers to the linearity of the graded Poisson bracket, while the third equality is the Jacobi identity. For simplicity, we will henceforth refer to the graded Poisson bracket simply as Poisson brackets.

With the choice of $F = \Pi^B(\vec{x}, t)$ and $G = \Phi_A(\vec{y}, t)$ in Eq. (2.12), we recover the canonical relation between the fields and their momenta

$$\left[\Pi^B(\vec{x}, t), \Phi_A(\vec{y}, t)\right]_P = -\delta_A^B \delta(x, y) \, . \tag{2.14}$$

The time evolution of any functional of the fields is determined from its Poisson bracket with the Hamiltonian

$$\dot{F} = [F, H_C]_P \ . \tag{2.15}$$

The Hamiltonian formulation given above is incomplete when the mapping from velocities to momenta given in Eq. (2.8) cannot be used to solve for all the velocities in terms of their canonically conjugate momenta. The Hamiltonian in such theories can be constructed through the Dirac-Bergmann formalism, which will be reviewed in the following section.

2.2 Constrained Field Theories

The Hamiltonian in Eq. (2.11) provides a complete description of the dynamics of the system only if all velocities of the theory uniquely map into the momenta. The momenta defined in Eq. (2.8) fail to be independent in systems where the following matrix, called the Hessian, is degenerate

$$W^{AB} = \frac{\delta^2 L}{\delta\dot{\Phi}_A \delta\dot{\Phi}_B} \, . \tag{2.16}$$

The Lagrangian of such theories is said to be singular and the theory possesses constraints involving the velocities and momenta of the theory. Some of these constraints which follow directly from the Lagrangian of the theory can be deduced from the nullity of the Hessian and are called primary constraints. Assuming then that there are M primary constraints, we denote them by

$$P_m(\Phi_A(x), \Pi^A(x)) \approx 0, \qquad m = 1, \cdots, M \, ; \, M < 2N \, . \tag{2.17}$$

The constraints are satisfied on a $2N - M$ infinite-dimensional subspace of phase space, which we will call the 'constraint subspace'. The symbol $\approx$ in Eq. (2.17) stands for 'weakly equal', meaning 'equal on the constraint subspace'. In other words, two quantities on the phase space are weakly equal only if they differ by a linear combination of the constraints. While the constraints vanish on the subspace, their variations need not. Thus in particular, Poisson brackets must first be evaluated before setting the constraints to vanish. An equality is said to be strongly equal if it holds throughout phase space and not just on the constraint subspace.

The Dirac-Bergmann formalism provides a step-by-step procedure to determine all the constraints of the theory. We first include the constraints in Eq. (2.17) to the canonical Hamiltonian given in Eq. (2.11) and define

$$\tilde{H} = \int dV_x \, (\Pi^A \dot{\Phi}_A + v^m P_m) - L \approx H_C \,, \tag{2.18}$$

where the v^m denote Lagrange multipliers, which are arbitrary functions of the canonical variables and coordinates. Consistency requires that these constraints be respected in time, i.e.

$$\dot{P}_m = \left[P_m, \tilde{H}\right]_P \approx 0 \,. \tag{2.19}$$

This may be satisfied in broadly one of two ways. If $\left[P_m, \tilde{H}\right]_P$ is a linear combination of existing constraints, $\dot{P}_m \approx 0$ automatically. If the Poisson bracket does not vanish, the requirement that $\left[P_m, \tilde{H}\right]_P \approx 0$ either leads to conditions which the Lagrange multipliers v^m must satisfy weakly, or to new constraints Q^P . These new constraints are now added to the Hamiltonian in Eq. (2.18) with their own Lagrange multipliers and the constraint subspace is further restricted by vanishing Q^P. We now need to ensure that $\dot{Q}^P \approx 0$. The process just described is repeated until no further constraints result. The final Hamiltonian so derived will be called the total Hamiltonian, which we will denote as H_T. At the end of the process, let there be $P = 1, \cdots P < 2N - M$ constraints Q_P derived by requiring the consistency condition. These are called the secondary constraints of the theory.

Let us suppose that there are a total of $K(< 2N)$ constraints at the end of this procedure. Within the Hamiltonian formalism it is meaningful to classify these not as primary and secondary constraints, but rather as first-class and second-class constraints. A dynamical function is called first-class if it has weakly vanishing Poisson brackets with all the constraints of the theory, else the function is second-class. In particular, constraints which have weakly vanishing Poisson brackets with all other constraints are first-class constraints. Let us thus suppose that the K constraints are comprised of I first-class constraints, $\Omega_a\,; a, b, \cdots = 1, \cdots, I$, and J second-class constraints $S_\alpha\,; \alpha\,\beta, \cdots = 1, \cdots, J$. We will also assume that we can identify that part of the Hamiltonian which is independent of the constraints, which we denote by H_0. Then the total Hamiltonian H_T can be expressed as

$$H_T = \int_\Sigma dV_x \;\left(H_0 + v^a \Omega_a + v^\alpha S_\alpha\right) \,. \tag{2.20}$$

2.2.1 Second-Class Constraints and Dirac brackets

It follows from the definition of second-class functions that the Poisson bracket of the second-class constraints among themselves do not weakly vanish. Hence transformations generated by second-class constraints could map physical configurations of the fields to unphysical ones. The elimination of these constraints can be achieved through the construction of Dirac brackets, which are defined as

$$[F,\, G]_D = [F,\, G]_P - \int_\Sigma dV_z \int_\Sigma dV_w \;\left[F, S_\alpha(z)\right]_P \, C^{-1}_{\alpha\beta}(z,w)\; \left[S_\beta(w), G\right]_P \,, \tag{2.21}$$

where $C^{-1}_{\alpha\beta}$ is the inverse of the matrix built from the Poisson brackets of second-class constraints

$$C_{\alpha\beta}(x,y) = \left[S_\alpha(x), S_\beta(y)\right]_P \,. \tag{2.22}$$

The Dirac brackets satisfy the relations given in Eq. (2.13), if the Poisson brackets are replaced by Dirac brackets. Further, the Dirac brackets vanish, by construction, should any of its arguments be a second-class constraint, i.e. $[S_\alpha, A]_D = 0$, with A denoting any functional of the fields. After implementing the Dirac brackets we can thus set the second-class constraints to vanish. As a result, the theory now comprises of a total Hamiltonian which do not involve the second-class constraints, whose dynamics are governed by Dirac brackets.

2.2.2 First-Class Constraints and Gauge Fixing

An important property of first-class functions is that they preserve the Poisson bracket structure. Let F and G denote two first-class functions and $\Psi_i\,; i, j, k, \cdots = 1, \cdots K$ denote all the constraints of the theory. The first-class property implies that the Poisson brackets $[F\,, \Psi_k]_P$ and $[G\,, \Psi_k]_P$ can be expressed as a linear combination of the constraints Ψ_k. It then follows from the Jacobi identity that $[[F\,, G]_P\,, \Psi_k]_P$ is also a linear combination of the constraints, and therefore weakly vanishes. Thus the Poisson bracket of two first-class functions is also first-class and preserves all the existing constraints of the theory.

The first-class constraints in addition generate gauge transformations. For simplicity, let us assume that all the constraints appearing in the total Hamiltonian of Eq. (2.20) involve only first-class constraints. Then the variation of an arbitrary functional B of the fields in a time interval δt is given by

$$\delta B = \delta t \;\left([B\,, H_0]_P + v^a\,[B\,, \Omega_a]_P\right) \tag{2.23}$$

However, the multipliers have an arbitrary dependence on the phase space variables, as well as time. By assuming that the values of the multipliers at the initial and final times are v^a and v'^a respectively, we see that the difference in the time evolution of the functional B due to the variation of the multipliers is given by

$$\Delta B = \Delta v^a \, [B, \Omega_a]_P \ , \tag{2.24}$$

where $\Delta v^a = \delta t(v^a - v'^a)$. Since Hamilton's equations fully determine the final configuration of the system for any given initial configuration, the variation in Eq. (2.24) must be physically irrelevant. Thus the infinitesimal contact transformation in Eq. (2.24) represents a gauge transformation generated by $\Delta v^a \Omega_a$, i.e. the first-class constraints of the theory.

2.3 Discussion

Following the elimination of second-class constraints, it is also desirable to further eliminate the first-class constraints and thereby the gauge redundancy of the theory completely. One approach involves solving the first-class constraints directly, thereby reducing the number of independent phase space variables. A more systematic approach is based on the introduction of 'gauge-fixing' constraints, which are not derived from either the Lagrangian or the Hamiltonian, but which have non-vanishing Poisson brackets with the first-class constraints. In this manner, each first-class constraint is replaced by two second-class constraints. These constraints can now be eliminated by constructing the Dirac brackets of the gauge fixed theory.

However, this approach works insofar as Dirac brackets can be defined. General constrained systems may provide a matrix of the Poisson brackets of second-class constraints whose inverse cannot be exactly determined. For example, in the case of the Yang-Mills field, the inverse of the matrix needed to define the Dirac brackets can only be solved perturbatively. The resulting field dependence in the Dirac brackets obstructs the usual canonical quantization of the theory [81]. In such cases, the

Hamiltonian BRST formalism for constrained theories can prove useful [79]. Aspects of the Hamiltonian BRST formalism relevant to this thesis and its formulation on spherically symmetric backgrounds with horizons will be considered in Chapter 5.

3 Constrained dynamics on spherically symmetric backgrounds

In this chapter, I will apply the Dirac-Bergmann formalism to field theories on static, spherically symmetric black hole backgrounds, which may either be asymptotically flat or possess a cosmological horizon. These spacetimes are endowed with a timelike Killing vector field, whose norm vanishes on the horizons of the background. This allows us to consider spatial hypersurfaces with boundaries corresponding to the spatial sections of the horizon(s) of the spacetime. By considering the time evolution with respect to the Killing vector field, the background and the horizons are fixed while the fields defined on them evolve in time. The first section of this chapter describes the foliation of the spacetime and the definition of time derivatives, which will be used to investigate the Hamiltonian dynamics of field theories.

The constraints of field theories often involve the derivatives of fields. Thus the presence of boundaries on the hypersurface could modify the constraints. Since the constraints are also specific to a given theory, we will consider the effect of horizons on the constraints through two examples – the Maxwell field and the Abelian Higgs model. In both cases, we find that the Gauss law constraint now involve additional surface terms due to the horizons of the spacetime. The modified Gauss law constraint leads to certain implications on the observed charge. We first show that any surface whose radius is greater than the black hole horizon has a non-vanishing flux through it, thereby enclosing a non-vanishing charge. We however also demonstrate that this flux vanishes in taking the limit of this surface to the event horizon of the black hole. This suggests that a black hole horizon might act as a dipole layer, comprising of opposite charges on either side of the horizon. While the charges on one side of the horizon are screened for an external observer, this is not the case for an observer exactly at the horizon.

We will also consider the gauge transformations and gauge fixing of the theory in light of the modified constraints. We will find that the gauge transformations of both the Maxwell field and the Abelian Higgs model take the same form as on curved backgrounds without boundaries. We then adopt the usual radiation gauge for the Maxwell field and the unitary gauge for the Abelian Higgs field. This leads to covariant extensions of the known Dirac brackets for these theories. When the radiation gauge is adopted for the Maxwell field, the Dirac brackets involve the Green function for the spacetime Laplacian operator. Using the expression for this Green function on the Schwarzschild background, we show that the Dirac bracket reduces to the Poisson bracket when any one of its arguments is considered at the event horizon. We have also considered a modified radiation gauge involving an additional surface term, which serves to fix the fields at the horizon. In this case, the resulting Dirac brackets involve the Green function for the spatial Laplacian of the hypersurface. The expression of this Green function about the Schwarzschild background has been derived in the appendix of this chapter. Using this expression, we show that the Dirac brackets in the radiation gauge which involves surface terms at the horizon of a Schwarzschild black hole, remain distinct from Poisson brackets when any one of its arguments is evaluated at the horizon.

3.1 Foliation of the background

We will consider static, spherically symmetric, torsion-free spacetimes endowed with at least one horizon. Thus there exists a timelike Killing vector field ξ^a satisfying

$$\nabla_{(a}\xi_{b)} = 0\,, \tag{3.1}$$

which is normalized such that $\lambda^2 = -\xi^a\xi_a$, with $\lambda = 0$ at the horizons. This Killing vector satisfies the Frobenius theorem and hence

$$\xi_{[a}\nabla_b\xi_{c]} = 0\,. \tag{3.2}$$

It follows that there exists a one-parameter family of integrable, spacelike hypersurfaces Σ which are everywhere orthogonal to ξ^a . The background may have one or more Killing horizons, which are surfaces where $\lambda = 0$. Spatial sections of the Killing horizons $\mathcal{H}$ are submanifolds of Σ. Thus the region of spacetime under consideration is $\mathcal{H} \cup \pm \cup \gamma_\xi$, where γ_ξ represents the timelike orbits of ξ^a. For an asymptotically flat or anti-de Sitter spacetime, this represents the region 'outside the horizon'. For spacetimes with a positive cosmological constant, for example a static de Sitter black hole spacetime, the region under consideration is 'between the horizons'. The induced metric on Σ is given by

$$h_{ab} = g_{ab} + \lambda^{-2}\xi_a\xi_b\,. \tag{3.3}$$

From Eq. (3.3), it also follows that the determinant of the spacetime metric satisfies

$$\sqrt{-g} = \lambda\,\sqrt{h}\,. \tag{3.4}$$

The spatial sections of the Killing horizons $\mathcal{H}$ of the background are closed, spherically symmetric surfaces. As these surfaces are submanifolds of Σ, the induced metric on $\mathcal{H}$ can be written as

$$\sigma_{ab} = h_{ab} - n_a n_b\,, \tag{3.5}$$

where n^a is a unit spatial normal, $n_a n^a = 1$, which points in the direction of increasing time.

We can define the projection operator h^a_b

$$h^a_b = \delta^a_b + \lambda^{-2}\xi^a\xi_b\,, \tag{3.6}$$

which projects spacetime tensors onto the spatial hypersurface Σ. We also define a covariant derivative $\mathcal{D}_a$ compatible with the metric h_{ab} of the hypersurface Σ

$$\mathcal{D}_b = h^a_b \nabla_b \quad ; \quad \mathcal{D}_a h_{bc} = 0\,. \tag{3.7}$$

Using Eq. (3.6), we have by definition the following projection on Σ

$$\begin{aligned} h^a_g \cdots h^b_f h^l_c \cdots h^m_d T^{g\cdots f}_{l\cdots m} &= t^{a\cdots b}_{c\cdots d}\,, \\ h^n_e h^a_g \cdots h^b_f h^l_c \cdots h^m_d \nabla_n T^{g\cdots f}_{l\cdots m} &= \mathcal{D}_e t^{a\cdots b}_{c\cdots d}\,, \end{aligned} \tag{3.8}$$

where ∇_a is the spacetime covariant derivative, $T^{a\cdots b}_{c\cdots d}$ represents a spacetime tensor and $t^{a\cdots b}_{c\cdots d}$ denotes its projection on Σ. The projected tensors in Eq. (3.8) are denoted with lowercase alphabets, which will be the convention followed in this thesis.

The time evolution of the fields will be generated by the Lie derivative with respect to the timelike Killing vector field ξ^a

$$\dot{T}^{a\cdots b}_{c\cdots d} = \pounds_\xi T^{a\cdots b}_{c\cdots d}\,. \tag{3.9}$$

Since $\pounds_\xi g_{ab} = 0 = \pounds_\xi \xi^a$, it follows that $\pounds_\xi h_{ab} = 0$. Thus time evolution and the projection of tensors can be performed on any spacetime tensor in any order.

We will now consider the projection of a given action, which will be needed to define the canonical Hamiltonian. Let $\Psi^A\,, A = 1, \cdots N$, define fields on the spacetime. The action involving the fields and their derivatives is given by the volume integral of the Lagrangian density over the spacetime manifold $\mathcal{M}$

$$S = \int dV_4^x\ \mathcal{L}(\Psi_A(x), \nabla_a \Psi_A(x))\,. \tag{3.10}$$

From Eq. (3.4), the covariant volume element can be expressed as $dV_4^x = \lambda(x)dtdV_x$. Denoting the projection of the fields Ψ^A by Φ^A, we can then always determine the projected action

$$S = \int dt \int_\Sigma dV_x \ \lambda(x)\mathcal{L}(\Phi_A(x), \mathcal{D}_a\Phi_A(x), \pounds_\xi\Phi_A(x))$$
$$= \int dt \ L(\Phi_A(x), \mathcal{D}_a\Phi_A(x), \pounds_\xi\Phi_A(x)) \,, \tag{3.11}$$

where we have defined the Lagrangian L as the volume integral of the Lagrangian density over Σ in Eq. (3.11). The treatment now follows Chapter 2, with the foliation of the background as defined above. The definition of the momenta are now as given in Eq. (2.8)

$$\Pi^A = \frac{\delta L}{\delta \dot{\Phi}_A} \,. \tag{3.12}$$

As in Eq. (2.9), Eq. (3.12) involves an 'equal-time' functional derivative on the hypersurface, which satisfies

$$\frac{\delta\Phi_A(\vec{x},t)}{\delta\Phi_B(\vec{y},t)} = \delta_A^B \, \delta(x,y) \,, \tag{3.13}$$

where the $\delta(x,y)$ is the covariant delta function on Σ defined in Eq. (2.10) which satisfies

$$\int_\Sigma dV_y \ \delta(x,y) f(\vec{y},t) = f(\vec{x},t) \,. \tag{3.14}$$

The momenta in Eq. (3.12) define the canonical Hamiltonian through the Legendre transform

$$H_C = \int_\Sigma dV_x \ \Pi^A \dot{\Phi}_A - L \,. \tag{3.15}$$

The Dirac-Bergmann formalism can now be carried out exactly as described in the previous chapter. However, as mentioned in the introduction of this chapter, we expect that the 'boundaries' of the hypersurface (corresponding to the horizons of the spacetime) could affect the constraints of field theories known in the absence of boundaries. In the next sections, we will consider the Maxwell field and the Abelian Higgs model, where we will investigate the effect of the horizons of the background on these theories.

3.2 The Maxwell field

The covariant action for the Maxwell field is given by

$$S_{EM} = \int dV_4^x \left(-\tfrac{1}{4}F_{ab}F_{cd}g^{ac}g^{bd}\right) , \tag{3.16}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ and dV_4 is the four-dimensional volume form on the manifold $\mathcal{M} = \Sigma \times \mathbb{R}$. Using $dV_4^x = \lambda \, dt \, dV_x$ and defining $a_b = h_b^a A_a$, $\phi = A_a \xi^a$, $e_a = -\lambda^{-1}\xi^c F_{cd}$ and $f_{ab} = h_a^c h_b^d F_{cd}$, we find the following projected action

$$S_{EM} = -\int dt \int_\Sigma dV_x \frac{\lambda}{4}\left(f_{ab}f^{ab} - 2e_a e^a\right) = \int dt \; L_{EM} \, . \tag{3.17}$$

From Eq. (3.9), the time derivative of the field A_a can be expressed as

$$\begin{aligned} \dot{A}_b \equiv \pounds_\xi A_b &= \xi^a \nabla_a A_b + A_a \nabla_a \xi^a \\ &= \xi^a F_{ab} + \nabla_a (A_b \xi^b) \, . \end{aligned} \tag{3.18}$$

By projecting this expression, we find

$$\dot{a}_b = -\lambda e_b + \mathcal{D}_b \phi \, . \tag{3.19}$$

Since the velocity term $\pounds_\xi \phi$ does not appear in the electromagnetic Lagrangian Eq. (3.17), it implies that the momentum conjugate to ϕ vanishes and is a constraint of the theory

$$\frac{\delta L_{EM}}{\delta \dot{\phi}} = \pi^\phi = 0 \, . \tag{3.20}$$

The momenta corresponding to a_b are given by

$$\pi^b = \frac{\delta L_{EM}}{\delta \dot{a}_b} = -e^b \,. \tag{3.21}$$

The canonical Hamiltonian now follows from the Legendre transform

$$\begin{aligned} H_C &= \int_\Sigma dV_x \, \left(\pi^b \dot{a}_b\right) - L \\ &= \int_\Sigma dV_x \, \left(\lambda \left(\frac{1}{2}\pi^b \pi_b + \frac{1}{4} f_{ab} f^{ab}\right) + \pi^b \mathcal{D}_b \phi\right) \,. \end{aligned} \tag{3.22}$$

We add the primary constraint to Eq. (3.22) to define a new Hamiltonian $\tilde{H}$

$$\tilde{H} = \int_\Sigma dV_x \, \left(\lambda \left(\frac{1}{2}\pi^b \pi_b + \frac{1}{4} f_{ab} f^{ab}\right) + \pi^b \mathcal{D}_b \phi + v_\phi \pi^\phi\right) \,, \tag{3.23}$$

where v_ϕ is a Lagrange multiplier. The canonical Poisson brackets of the theory are given by

$$\begin{aligned} \left[\phi(x), \pi^\phi(y)\right]_P &= \delta(x,y) \,, \\ \left[a_a(x), \pi^b(y)\right]_P &= \delta_a^b \delta(x,y) \,, \end{aligned} \tag{3.24}$$

where the covariant delta function $\delta(x,y)$ satisfies Eq. (3.14).

3.2.1 The Dirac-Bergmann formalism

We will now apply the Dirac-Bergmann formalism to determine all the constraints of the theory. In order to find any additional constraints, we need to ensure that the existing constraints are satisfied at all times. Since the constraints are distribution-valued functions of phase space, we will require the use of smearing (test) functions to evaluate the Poisson bracket. We thus introduce a non-dynamical smearing function

ϵ, using which we will calculate $\epsilon\dot{\pi}^\phi = \left[\epsilon\pi^\phi, \tilde{H}\right]_P$. This Poisson bracket is calculated as follows

$$\begin{aligned}\int_\Sigma dV_y \epsilon(y)\dot{\pi}^\phi(y) &= \int_\Sigma dV_y \epsilon(y) \left[\pi^\phi(y), \tilde{H}\right]_P \\ &= \int_\Sigma dV_y \epsilon(y) \left[\pi^\phi(y), \int_\Sigma dV_x \pi^b(x)\mathcal{D}^x_b\phi(x)\right]_P \\ &= -\oint_{\partial\Sigma} da_y\, \epsilon(y) n^y_b \pi^b(y) + \int_\Sigma dV_y\, \epsilon(y)\left(\mathcal{D}^y_b\pi^b(y)\right) . \end{aligned} \tag{3.25}$$

In deriving this result, we have used the canonical Poisson brackets given in Eq. (3.24) and an integration by parts. The smearing function ϵ is assumed to be well behaved, but we make no further assumption regarding its properties. In particular we do not assume that ϵ or its derivatives vanish on the horizons of the background.

The surface integral in Eq. (3.25) is to be considered as a sum over all the surfaces on the background. Thus, for black hole backgrounds with a cosmological horizon, there are in fact two surface integrals. The unit normal n^a would then denote the outward pointing normal at the black hole horizon and the inward pointing normal at the cosmological horizon. The area element at the horizons is finite and we have already assumed that ϵ is finite there. By using the Schwarz inequality, we find that the remaining terms in the surface integrand are also finite

$$\left|n_b\pi^b\right| \leq \sqrt{|n_b n^b|\ |\pi_b\pi^b|} . \tag{3.26}$$

In this expression, $n_b n^b = 1$ by definition since n_b is the unit spatial normal to the horizon, and $\pi_b\pi^b = e_b e^b$ appears in the energy momentum tensor (more precisely in invariant scalars such as $T^{ab}T_{ab}$), and therefore is finite at the horizon. Thus the integral over $\partial\Sigma$ is finite and provides a non-vanishing contribution from the horizons to the constraint. Thus Eq. (3.25) provides the following integrated expression for the smeared constraint

$$\int_\Sigma dV_y\,\epsilon(y)\Omega_2(y) = -\oint_{\partial\Sigma} da_y\,\epsilon(y)n_b^y\pi^b(y) + \int_\Sigma dV_y\,\epsilon(y)\mathcal{D}_b^y\pi^b(y)\,. \tag{3.27}$$

Apart from the smearing function, the integrand in Eq. (3.27) can be written as the following constraint comprising of a bulk and a surface term

$$\Omega_2 = -n_b\pi^b\Big|_{\mathcal{H}} + \mathcal{D}_b\pi^b \approx 0\,. \tag{3.28}$$

The '$|_{\mathcal{H}}$' denotes the surface term contribution from the horizon(s). Eq. (3.28) represents an expression which must always be smeared and integrated. By using a smearing function ϵ which is regular at the horizons, the integration of Eq. (3.28) is as given in Eq. (3.27).

We will now show that there are no further constraints resulting from $\dot{\Omega}_2 \approx 0$. We first include the new constraint with a multiplier into the existing Hamiltonian given in Eq. (3.23), which gives us

$$H_T = \tilde{H} + \int_\Sigma dV_x\;v_1\mathcal{D}_b\pi^b - \oint_{\partial\Sigma} da_x\;v_1 n_b\pi^b\,. \tag{3.29}$$

$\dot{\Omega}_2$ now follows by evaluating the Poisson bracket of the smeared constraint with the new Hamiltonian. We find

$$\begin{aligned}
\int_\Sigma dV_y\,\epsilon(y)\dot{\Omega}_2(y) &= \int_\Sigma dV_y\,\epsilon(y)\left[\Omega_2(y), H_T\right]_P \\
&= -\int_\Sigma dV_y\mathcal{D}_b^y\left(\epsilon(y)\right)\int_\Sigma dV_x\left[\pi^b(y), \mathcal{D}_a^x a_c(x)\right]_P f^{ac}(x) \\
&= \int_\Sigma dV_y\mathcal{D}_a^y\mathcal{D}_b^y\epsilon(y)f^{ab}(y) \\
&= 0\,.
\end{aligned} \tag{3.30}$$

The last equality follows from the antisymmetry of f^{ac} in its indices, where we have used the fact that $\mathcal{D}_a$ is torsion-free. Thus the total Hamiltonian is that of Eq. (3.29) with the expression

$$H_T = \int\limits_\Sigma dV_x \left(\lambda \left(\frac{1}{4} f_{ab} f^{ab} + \frac{1}{2} \pi_a \pi^a \right) + v_1 \mathcal{D}_b \pi^b + \pi^b \mathcal{D}_b \phi + v_\phi \pi^\phi \right) - \oint\limits_{\partial\Sigma} da_x \, n_b v_1 \pi^b \tag{3.31}$$

The multipliers v_1 and v_ϕ can be determined from the equations of motion. The evolution of ϕ is given by

$$\begin{aligned} \int\limits_\Sigma dV_y \epsilon(y) \dot{\phi}(y) &= \int\limits_\Sigma dV_y \epsilon(y) \left[\phi(y), H_T \right]_P \\ &= \int\limits_\Sigma dV_y \epsilon(y) \int\limits_\Sigma dV_x v_\phi(x) \left[\phi(y), \pi^\phi(x) \right]_P \\ &= \int\limits_\Sigma dV_y \epsilon(y) v_\phi(y) \, , \end{aligned} \tag{3.32}$$

which tells us that $v_\phi = \dot{\phi}$. The evolution of a_b gives us

$$\begin{aligned} \int\limits_\Sigma dV_y \, \epsilon(y) \dot{a}_b(y) &= \int\limits_\Sigma dV_y \, \left[\epsilon(y) a_b(y), H_T \right]_P \\ &= \int\limits_\Sigma dV_y \, \epsilon(y) \int\limits_\Sigma dV_x \left[a_b(y), \pi^c(x) \right]_P \left(\lambda(x) \pi_c(x) + \mathcal{D}^x_c \phi(x) - \mathcal{D}^x_c v_1(x) \right) \\ &= \int\limits_\Sigma dV_y \epsilon(y) \left[\lambda(y) \pi_b(y) + \mathcal{D}^y_b \phi(y) - \mathcal{D}^y_b v_1(y) \right] \, . \end{aligned} \tag{3.33}$$

Comparing this with the expression for $\dot{a}_b$ in Eq. (3.19), we find that $\mathcal{D}_b v_1 = 0$. While v_1 can be any constant, we will for simplicity assume that $v_1 = 0$. With these values for the multipliers, the total Hamiltonian in Eq. (3.31) takes the form

$$H_T = \int\limits_\Sigma dV_x \left(\lambda \left(\frac{1}{4} f_{ab} f^{ab} + \frac{1}{2} \pi_a \pi^a \right) + \pi^b \mathcal{D}_b \phi + \dot{\phi} \pi^\phi \right) \, . \tag{3.34}$$

3.2.2 Gauge transformations and Gauge fixing

The two constraints of the Maxwell field are both first class and therefore generate local gauge transformations of the fields. To determine these transformations, we construct the general linear combination of the constraints

$$\Delta(x) = \int_{\Sigma} dV_x \alpha_1(x)\Omega_1(x) + \alpha_2(x)\Omega_2(x)\,, \tag{3.35}$$

where $\Omega_1 = \pi^\phi$, $\Omega_2 = \mathcal{D}_a\pi^a - n_a\pi^a\Big|_{\mathcal{H}}$ and $\alpha_1\,,\alpha_2$ are two arbitrary differentiable functions. We now find the following non-vanishing Poisson brackets of Δ with the fields

$$\begin{aligned}\delta_1\phi(x) &= [\phi(x),\Delta(y)]_P = \alpha_1(x)\\ \delta_2 a_b(x) &= [a_b(x),\Delta(y)]_P = -\mathcal{D}^x_b\alpha_2(x)\end{aligned} \tag{3.36}$$

These transformations can be identified with the usual gauge transformations $A_\mu \to A_\mu + \partial_\mu\alpha$ of the covariant Lagrangian in Eq. (3.16) if we identify $\alpha_2(y) = -\alpha(y)$ and $\alpha_1(y) = \pounds_\xi\alpha(y)$. We note that the gauge transformations for this background are the same as those on backgrounds without horizons. The surface terms in the Gauss law constraint ensure that the gauge transformations of a_b remain unaltered. We also note that $a_b(x)$ and $\phi(x)$ in Eq. (3.36) could be located anywhere on Σ, including at the horizons. This can equivalently be seen as a result of the functions α_1 and α_2 being regular at the horizons, which distinguishes our result for gauge transformations at horizons from those on spatial boundaries.

To proceed further, we take the approach of converting the gauge constraints into second class ones by fixing the gauge. We will first adopt the radiation gauge in the following and later on, a modified radiation gauge which also involves a surface term. The radiation gauge is given by

$$\begin{aligned}
\Omega_1 &= \pi^\phi \\
\Omega_2 &= \mathcal{D}_a \pi^a - n_a \pi^a \Big|_{\mathcal{H}} \\
\Omega_3 &= \phi \,, \\
\Omega_4 &= \mathcal{D}^b \left(\lambda a_b\right) \,.
\end{aligned} \tag{3.37}$$

This gauge involves the constraints $\Omega_3 \approx 0$ and $\Omega_4 \approx 0$, in addition to the two first-class constraints of the theory, $\Omega_1 \approx 0$ and $\Omega_2 \approx 0$. Collectively, the constraints in Eq. (3.37) are second-class and have the following non-vanishing Poisson brackets

$$\begin{aligned}
\left[\Omega_1(x), \Omega_3(y)\right]_P &= -\delta(x,y) \,, \\
\left[\Omega_2(x), \Omega_4(y)\right]_P &= \mathcal{D}^y_a \left(\lambda(y) \mathcal{D}^a_y \delta(x,y)\right) \,.
\end{aligned} \tag{3.38}$$

The first Poisson bracket in Eq. (3.38) results directly from the canonical relations. We evaluate the second Poisson bracket using two smearing functions $\epsilon(y)$ and $\gamma(x)$ which are regular at the horizons as follows

$$\begin{aligned}
&\left[\int_\Sigma dV_x \; \gamma(x)\Omega_2(x), \int_\Sigma dV_y \; \epsilon(y)\Omega_4(y)\right]_P \\
&\quad = \left[\int_\Sigma dV_x \left(\mathcal{D}^x_a \gamma(x)\right) \pi^a(x) \,, \int_\Sigma dV_y \left(\mathcal{D}^b_y \epsilon(y)\right) \lambda(y) a_b(y)\right]_P \\
&\quad = -\int_\Sigma dV_y \; \lambda(y) \left(\mathcal{D}^y_a \gamma(y)\right) \left(\mathcal{D}^a_y \epsilon(y)\right) \\
&\quad = -\oint_{\partial\Sigma} da_y \; \lambda(y)\epsilon(y) n^a_y \left(\mathcal{D}^y_a \gamma(y)\right) + \int_\Sigma dV_y \; \epsilon(y) \mathcal{D}^a_y \left(\lambda(y) \mathcal{D}^y_a \gamma(y)\right) .
\end{aligned} \tag{3.39}$$

The first and last equality in Eq. (3.39) result from an integration by parts. Since the smearing functions ϵ and γ are regular at the horizons, the surface integral in the last equality of Eq. (3.39) vanishes on account of the Schwarz inequality

$$\begin{aligned} \left|\lambda n^a D_a\left(\gamma\right)\right|^2 &\leq \lambda^2 \left|n^a n_a\right| \left|h^{ab}\left(D_a\gamma\right)\left(D_b\gamma\right)\right| \\ &= 0 \qquad \text{(at the horizons)}\,. \end{aligned} \tag{3.40}$$

Thus the only contribution of Eq. (3.39) comes from the volume term which, upon using the definition of the delta function in Eq. (3.14), can be rewritten as

$$\left[\int\limits_{\Sigma} dV_x\ \gamma(x)\Omega_2(x), \int\limits_{\Sigma} dV_y\ \epsilon(y)\Omega_4(y)\right]_P = \int\limits_{\Sigma} dV_y\, \epsilon(y) \int\limits_{\Sigma} dV_x\ \gamma(x)\left[\mathcal{D}_y^a\left(\lambda(y)\mathcal{D}_a^y\left(\delta(x,y)\right)\right)\right]\,. \tag{3.41}$$

Hence the Poisson brackets between the constraints are those given in Eq. (3.38). The matrix of the Poisson brackets between these constraints have a non-vanishing determinant and is invertible. This matrix $C_{\alpha\beta}\left(x,y\right) = \left[\Omega_\alpha(x), \Omega_\beta(y)\right]_P$ is given by

$$C(x,y) = \begin{pmatrix} 0 & 0 & -\delta(x,y) & 0 \\ 0 & 0 & 0 & \mathcal{D}_a^y\left(\lambda(y)\mathcal{D}_y^a\delta(x,y)\right) \\ \delta(x,y) & 0 & 0 & 0 \\ 0 & -\mathcal{D}_a^y\left(\lambda(y)\mathcal{D}_y^a\delta(x,y)\right) & 0 & 0 \end{pmatrix}. \tag{3.42}$$

The Dirac brackets for two phase space functionals A and B follows from Eq. (2.21)

$$\left[A\,,\,B\right]_D = \left[A\,,\,B\right]_P - \int\limits_{\Sigma} dV_u \int\limits_{\Sigma} dV_v\left[A\,,\,\Omega_a(u)\right]_P C_{ab}^{-1}\left(u,v\right)\left[\Omega_b(v)\,,\,B\right]_P\,. \tag{3.43}$$

The inverse of the matrix C_{ab} requires the solution of the following equation

$$\mathcal{D}_a^y\left(\lambda(y)\mathcal{D}_y^a G\left(x,y\right)\right) = -\delta\left(x,y\right)\,. \tag{3.44}$$

We will now demonstrate that $G\left(x,y\right)$ represents the time-independent Green function of the spacetime Laplacian operator. Given a general rank p antisymmetric tensor $\Theta^{ab\ldots d}$ whose Lie derivative with respect to ξ vanishes, $\pounds_\xi \Theta^{ab\ldots d} = 0$, and whose projection on the hypersurface Σ is given by $h_a^{a'} h_b^{b'} \ldots h_d^{d'} \Theta^{ab\ldots d} = \theta^{a'b'\ldots d'}$, we have the following identity

$$\lambda \left(\nabla_a \Theta^{ab\ldots d}\right) h_b^{b'} \ldots h_d^{d'} = \mathcal{D}_a \left(\lambda \theta^{ab'\ldots d'}\right) . \tag{3.45}$$

This in particular tells us that the one form $\nabla_a^y G\,(x,y)$, for a time-independent scalar function $G\,(x,y)$, satisfies the following relation

$$\nabla_a^y \nabla_y^a G\,(x,y) = \lambda(y)^{-1} \mathcal{D}_a^y \left(\lambda(y) \mathcal{D}_y^a G\,(x,y)\right) . \tag{3.46}$$

Thus the Green function equation for the spacetime Laplacian

$$\nabla_a^y \nabla_y^a G\,(x,y) = -\lambda(y)^{-1} \delta\,(x,y) \ , \tag{3.47}$$

is equivalent to the following Green function equation on the hypersurface Σ

$$\mathcal{D}_a^y \left(\lambda(y) \mathcal{D}_y^a G\,(x,y)\right) = -\delta\,(x,y) \ , \tag{3.48}$$

which is precisely Eq. (3.44). Thus the inverse of the matrix in Eq. (3.42) is

$$C^{-1}(x,y) = \begin{pmatrix} 0 & 0 & \delta(x,y) & 0 \\ 0 & 0 & 0 & G\,(x,y) \\ -\delta(x,y) & 0 & 0 & 0 \\ 0 & -G\,(x,y) & 0 & 0 \end{pmatrix} . \tag{3.49}$$

Using this matrix in Eq. (3.43), we find the following non-vanishing Dirac bracket

$$\left[a_a(x), \pi^b(y)\right]_D = \delta(x,y)\delta_a^b - \mathcal{D}_a^x \left(\lambda(y) \mathcal{D}_y^b G\,(x,y)\right) . \tag{3.50}$$

This result follows from making no assumptions about $G\,(x,y)$ and its derivatives at the horizons. The Green function involved in the Dirac bracket of Eq. (3.50) has been derived about the Schwarzschild background in [88–91]. Its expression in spherical polar coordinates for two points $\vec{r} = (r\,,\theta\,,\phi)$ and $\vec{r}' = (r'\,,\theta'\,,\phi')$ on Σ is given by [91]

$$G\left(\vec{r}\,,\vec{r}'\right) = \frac{1}{rr'}\left[\frac{(r-m)(r'-m) - m^2\cos\gamma}{\sqrt{(r-m)^2 + (r'-m)^2 - 2(r-m)(r'-m)\cos\gamma - m^2\sin^2\gamma}} + m\right]\,, \tag{3.51}$$

where m denotes the mass of the black hole and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi - \phi'\right)$. Using the expression in Eq. (3.51), we find that the term $\mathcal{D}^x_a\left(\lambda(y)\mathcal{D}^b_y G\left(x,y\right)\right)$ appearing in Eq. (3.50) has a non-vanishing expression outside the horizon of the Schwarzschild black hole and vanishes when any one of its arguments, x or y, is at the horizon. Upon substituting Eq. (3.51) in Eq. (3.50), we find the following limit when $r' \to r_{\mathcal{H}}$ and $r \neq r_{\mathcal{H}}$

$$\left(\left[a_a(\vec{r}), \pi^b(\vec{r}')\right]_D\right)_{r' \to r_{\mathcal{H}}} = \delta(\vec{r}, \vec{r}_{\mathcal{H}})\delta^b_a = \left(\left[a_a(\vec{r}'), \pi^b(\vec{r})\right]_D\right)_{r' \to r_{\mathcal{H}}}\,, \tag{3.52}$$

where $\vec{r}_{\mathcal{H}} \equiv (r_{\mathcal{H}}, \theta, \phi)$ and $\left(\cdots\right)_{r' \to r_{\mathcal{H}}}$ denotes that we have taken the limit $r' \to r_{\mathcal{H}}$ of the argument in the brackets. Eq. (3.52) reveals that the Dirac bracket, when any one of its arguments is at the horizon of the Schwarzschild black hole, reduces to the canonical Poisson bracket between the fields. The Dirac bracket in the radiation gauge should only be non-vanishing for the transverse components of the Maxwell field, which is achieved through the Green function contribution in the Dirac bracket. We have just noted that this contribution vanishes at the horizon. This result in Eq. (3.52) can nevertheless be physically acceptable, as Killing horizons are actually null surfaces on which 'transverse' and 'longitudinal' lose meaning. However, since our analysis considers the *spatial section* of the horizon and not the horizon itself, we can seek a more appropriate gauge fixing choice which will provide Dirac brackets that are distinct from Poisson brackets even at the horizons of the spacetime.

This observation motivates us to adopt a radiation gauge which involves an additional surface term at the horizons of the background, analogous to the surface term in Ω_2 . This will allow us to consider how the horizon could affect the dynamics of the theory, which is what we primarily wish to explore in this thesis. Upon applying the gauge, we have the following four constraints

$$\begin{aligned}
\Omega_1 &= \pi^\phi \\
\Omega_2 &= \mathcal{D}_a \pi^a - n_a \pi^a \Big|_{\mathcal{H}} \\
\Omega_3 &= \phi \\
\Omega_4 &= \mathcal{D}^b a_b - n^b a_b \Big|_{\mathcal{H}} .
\end{aligned} \tag{3.53}$$

As in the usual radiation gauge, the following Poisson brackets are easily derived

$$\begin{aligned}
\left[\Omega_1(x), \Omega_3(y)\right]_P &= -\delta(x,y)\,, \\
\left[\Omega_2(x), \Omega_4(y)\right]_P &= \mathcal{D}_a \mathcal{D}^a \delta(x,y)\,.
\end{aligned} \tag{3.54}$$

The first Poisson bracket in Eq. (3.54) is simply one of the canonical relations. The second Poisson bracket is determined from the following calculation

$$\begin{aligned}
&\left[\int_\Sigma dV_x \gamma(x)\Omega_2(x), \int_\Sigma dV_y\ \epsilon(y)\Omega_4(y)\right]_P \\
&\qquad = \left[\int_\Sigma dV_x \left(\mathcal{D}^x_a \gamma(x)\right) \pi^a(x)\,, \int_\Sigma dV_y \left(\mathcal{D}^b_y \epsilon(y)\right) a_b(y)\right]_P \\
&\qquad = -\int_\Sigma dV_y \left(\mathcal{D}^y_a \gamma(y)\right) \left(\mathcal{D}^a_y \epsilon(y)\right) \\
&\qquad = -\oint_{\partial\Sigma} da_y\ \epsilon(y) n^a_y \left(\mathcal{D}^y_a \gamma(y)\right) + \int_\Sigma dV_y\ \epsilon(y) \mathcal{D}^a_y \mathcal{D}^y_a \gamma(y).
\end{aligned} \tag{3.55}$$

Using the Schwarz inequality, we find that the surface integrand satisfies

$$\begin{aligned}
\left|n^a D_a\left(\gamma\right)\right|^2 &\leq \left|n^a n_a\right| \left|h^{ab}\left(D_a \gamma\right)\left(D_b \gamma\right)\right| \\
&= h^{ab}\left(D_a \gamma\right)\left(D_b \gamma\right)\,.
\end{aligned} \tag{3.56}$$

The smearing functions and their derivatives are regular on the horizon, while $h^{rr} \sim \lambda^2$ on spherically symmetric backgrounds. Hence only the volume term of Eq. (3.55) contributes to the Poisson bracket. Using the definition of the delta function in Eq. (3.14), we can re-express the Poisson bracket as

$$\left[\int\limits_{\Sigma} dV_x\ \gamma(x)\Omega_2(x), \int\limits_{\Sigma} dV_y\ \epsilon(y)\Omega_4(y)\right]_P = \int\limits_{\Sigma} dV_y\ \epsilon(y) \int\limits_{\Sigma} dV_x\ \gamma(x) \left(\mathcal{D}^a_y \mathcal{D}^y_a \left(\delta(x,y)\right)\right) , \tag{3.57}$$

which is the Poisson bracket given in Eq. (3.54). The matrix of the Poisson brackets between these constraints, $C_{\alpha\beta}\left(x,y\right) = \left[\Omega_\alpha(x), \Omega_\beta(y)\right]_P$, is now given by

$$C(x,y) = \begin{pmatrix} 0 & 0 & -\delta(x,y) & 0 \\ 0 & 0 & 0 & \mathcal{D}^y_a \mathcal{D}^a_y \delta(x,y) \\ \delta(x,y) & 0 & 0 & 0 \\ 0 & -\mathcal{D}^y_a \mathcal{D}^a_y \delta(x,y) & 0 & 0 \end{pmatrix} . \tag{3.58}$$

Using Eq. (2.21), we have the definition of the Dirac bracket for two dynamical entities A and B

$$\left[A\,,\,B\right]_D = \left[A\,,\,B\right]_P - \int\limits_{\Sigma} dV_u \int\limits_{\Sigma} dV_v \left[A\,,\,\Omega_\alpha(u)\right]_P C^{-1}_{\alpha\beta}\left(u,v\right)\left[\Omega_\beta(v)\,,\,B\right]_P . \tag{3.59}$$

To evaluate the brackets, we now need to find the inverse of the operator $\mathcal{D}_a\mathcal{D}^a$. Let us formally write the inverse as $\widetilde{G}(x,y)$, i.e.

$$\mathcal{D}^y_a \mathcal{D}^a_y \widetilde{G}\left(x,y\right) = -\delta\left(x,y\right) , \tag{3.60}$$

for some scalar function $\widetilde{G}\left(x,y\right)$. This is the time-independent Green function for the spatial Laplacian operator of the hypersurface Σ. With this, the inverse matrix

$C^{-1}_{\alpha\beta}(x,y)$ can be written as

$$C^{-1}(x,y) = \begin{pmatrix} 0 & 0 & \delta(x,y) & 0 \\ 0 & 0 & 0 & \widetilde{G}\,(x,y) \\ -\delta(x,y) & 0 & 0 & 0 \\ 0 & -\widetilde{G}\,(x,y) & 0 & 0 \end{pmatrix} . \tag{3.61}$$

We can now substitute Eq. (3.61) in Eq. (3.43) to find the following non-vanishing Dirac bracket for the fields

$$\left[a_a(x), \pi^b(y)\right]_D = \delta(x,y)\delta^b_a - \mathcal{D}^x_a \mathcal{D}^b_y \widetilde{G}\,(x,y) \ . \tag{3.62}$$

Let us now consider how this bracket differs from that given in Eq. (3.50). The Green function for the spatial Laplacian operator on the Schwarzschild background has the following expression [84]

$$\begin{aligned}\widetilde{G}\,(\vec{r},\vec{r}') =& \frac{1}{\sqrt{rr'}}\left[\frac{\sqrt{\left(\kappa(r)r-m\right)\left(\kappa(r')r'-m\right)}}{\sqrt{\left(\kappa(r)r-m\right)^2+\left(\kappa(r')r'-m\right)^2-2\,\left(\kappa(r)r-m\right)\,\left(\kappa(r')r'-m\right)\,\cos\gamma}}\right.\\ &\left.+\frac{m\sqrt{\left(\kappa(r)r-m\right)\left(\kappa(r')r'-m\right)}}{\sqrt{\left(\kappa(r)r-m\right)^2\,\left(\kappa(r')r'-m\right)^2+m^4-2\,m^2\left(\kappa(r)r-m\right)\,\left(\kappa(r')r'-m\right)\,\cos\gamma}}\right] ,\end{aligned} \tag{3.63}$$

where $\kappa(r) = 1+\lambda(r) = 1+\sqrt{1-\frac{2m}{r}}$, m is the mass of the Schwarzschild black hole, $\vec{r} = (r,\theta,\phi)$, $\vec{r}' = (r',\theta',\phi')$ and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$ as before. A detailed derivation of this expression has been provided in the Appendix of this chapter. We can now substitute Eq. (3.63) in Eq. (3.62) and take the limit where any one of its arguments is at the horizon. In this case, we find the following Dirac bracket involving the radial component of the electric field π^r

$$\left[a_r(\vec{r}), \pi^r(\vec{y})\right]_D\Big|_{y\to r_{\mathcal{H}}} = \delta(\vec{r},\vec{r}_{\mathcal{H}}) + \kappa_{\mathcal{H}}\frac{2r-m(1+\cos\gamma)}{2\left(r^2-mr(1+\cos\gamma)\right)^{3/2}} = \left[a_r(\vec{y}), \pi^r(\vec{r})\right]_D\Big|_{y\to r_{\mathcal{H}}} , \tag{3.64}$$

where the constant $\kappa_{\mathcal{H}}$ in Eq. (3.64) denotes the surface gravity of the Schwarzschild black hole. The limit in Eq. (3.64) is quite different from the analogous limit found in Eq. (3.52) resulting from the radiation gauge without a surface term. While this result is gauge dependent, it does suggest that the quantization of fields could be affected by how fields are fixed at the horizon.

3.2.3 Charges

Given a Gauss law constraint, its volume integration provides the flux across a given Gaussian surface – the boundary of the region over which the volume integration is carried out. The Gauss law constraint thereby provides an expression for the charge contained within a region. On asymptotically flat black hole backgrounds it is has been usually assumed (on the basis of charge conservation) that since the charge contained in the spacetime can be expressed as a surface integral at infinity, a non-vanishing surface integral over the horizon of the black hole must exist and provides an expression for the charge contained behind the event horizon.

We will now consider the integration of the modified Gauss law constraint in Eq. (3.28) over a region of the hypersurface Σ_B, whose outer (spatial) boundary is located at r_B and whose inner radius is located at the event horizon r_H. The surfaces at r_B and r_H will be denoted by $\partial\Sigma_B$ and $\partial\Sigma_H$ respectively. In order to clarify the steps in this section, we will denote surface integrals which arise from integrating the divergence term $\mathcal{D}_b\pi^b$ by $\oint$ and the surface integral contributions due to the surface term of Eq. (3.28) by $\boldsymbol{\oint}$. The surface integrals have their usual meaning and the notation merely reflects where the surface integrals come from. We can now derive the expression for the charge contained within the region Σ_B by integrating Eq. (3.28) from r_H to r_B,where $r_B > r_H$. We find

$$\begin{aligned} Q_B &= \int\limits_{\Sigma_B} \Omega_2 \\ &= \oint\limits_{\partial\Sigma_B} n_b \pi^b - \oint\limits_{\partial\Sigma_H} n_b \pi^b + \oint\limits_{\boldsymbol{\partial\Sigma_H}} n_b \pi^b - \oint\limits_{\boldsymbol{\partial\Sigma_B}} n_b \pi^b \\ &= \oint\limits_{\partial\Sigma_B} n_b \pi^b \,. \end{aligned} \tag{3.65}$$

The last surface integral $\oint\limits_{\boldsymbol{\partial\Sigma_B}} n_b \pi^b$ in the second equality of Eq. (3.65) actually vanishes. This follows from Eq. (3.28), which tells us that the only surface terms which do contribute are those involving surface integrals over the horizons of the background. Thus only the first three terms in the second equality of Eq. (3.65) provide the result in the last line of Eq. (3.65). This is the usual expression for the charge as seen by an observer located at a radius r_B outside the horizon of the black hole. However, a crucial difference occurs if the above integral is considered in the limit where $\partial\Sigma_B \to \partial\Sigma_H$. In this case we find from Eq. (3.65)

$$\lim_{\partial\Sigma_B \to \partial\Sigma_H} Q_B = Q_H = 0 \,. \tag{3.66}$$

We note that in this case the fourth term in the second equality of Eq. (3.65) has a non-vanishing limit, leading to the result in Eq. (3.66). A similar result follows for backgrounds which possess a cosmological horizon, which we assume to be located at a radius r_C. When an outer horizon exists, we can't consider any Gaussian surface which encloses the cosmological horizon which lies within the hypersurface Σ. We can however always integrate entirely over Σ to find

$$
\begin{aligned}
Q &= \int\limits_{\Sigma} \Omega_2 \\
&= \oint\limits_{\partial\Sigma_C} n_b \pi^b - \oint\limits_{\partial\Sigma_H} n_b \pi^b + \oint\limits_{\boldsymbol{\partial\Sigma_H}} n_b \pi^b - \oint\limits_{\boldsymbol{\partial\Sigma_C}} n_b \pi^b \\
&= 0 \,.
\end{aligned}
\tag{3.67}
$$

Eq. (3.66) and Eq. (3.67) reveal an unexpected result, namely that a non-vanishing flux observed over a spatial surface outside the horizon of a black hole is seen to vanish as the surface approaches the horizon. Eq. (3.65), Eq. (3.66) and Eq. (3.67) suggest that the horizon may be viewed as a dipole layer, with the charge on one side of the horizon being screened from observation. For an observer outside the horizon, the black hole is a charged body which follows from the bulk contribution to the constraint. When the observer is at the horizon, the cancellation of "positive" and "negative" charges leads to the result given in Eq. (3.66). A similar argument holds for the cosmological horizon as seen in Eq. (3.67). To further resolve this observation, we will consider the Abelian Higgs model in the next section.

3.3 The Abelian Higgs model

The action we will now consider is given by

$$
S = -\int dV_4 \, \left(\frac{1}{4} g^{ac} g^{bd} F_{ab} F_{cd} + \frac{1}{2} g^{ab} \widetilde{\nabla}_a \Phi (\widetilde{\nabla}_b \Phi)^* + \frac{1}{4} \alpha \left(|\Phi|^2 - v^2 \right)^2 \right) \,, \tag{3.68}
$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the electromagnetic field strength, Φ is the Higgs field, $\widetilde{\nabla}_a \Phi = (\nabla_a + iqA_a)\Phi$ is the gauge covariant derivative and $\frac{1}{4}\alpha \left(|\Phi|^2 - v^2\right)^2$ with $\alpha > 0$ represents the Higgs potential. It will be convenient to parametrize the Higgs field as $\Phi = \rho e^{\frac{i\eta}{v}}$, following which the action becomes

$$S = -\int dV_4^x \, \left(\frac{1}{4} g^{ac} g^{bd} F_{ab} F_{cd} + \frac{1}{2} q^2 \rho^2 g^{ab} \left(A_a + \frac{1}{qv} \nabla_a \eta \right) \left(A_b + \frac{1}{qv} \nabla_b \eta \right) + \frac{1}{2} g^{ab} \nabla_a \rho \nabla_b \rho + \frac{1}{4} \alpha \left(\rho^2 - v^2 \right)^2 \right) . \tag{3.69}$$

A useful feature of the polar representation of the Higgs field Φ is that the magnitude ρ of the field is gauge invariant. Only the phase of the Higgs field η involves gauge transformations. In particular,the action is invariant under the local gauge transformations $A_a \to A_a + \nabla_a \alpha$ and $\eta \to \eta - vq\alpha$, where α is an arbitrary differentiable function. It is thus the combination $A_a + \frac{1}{qv} \nabla_a \eta$ which is gauge invariant.

By defining the projected quantities $a_a = h_a^b A_b$, $\phi = \xi^a A_a$, $f_{ab} = h_a^c h_b^d F_{cd}$ and $e_a = \lambda^{-1} \xi^b F_{ab}$, we find the projected action

$$S = -\int dt \int_\Sigma dV_x \; \lambda \left(\frac{1}{2} q^2 \rho^2 \left(h^{ab} \left(a_a + \frac{1}{qv} \mathcal{D}_a \eta \right) \left(a_b + \frac{1}{qv} \mathcal{D}_b \eta \right) - \lambda^{-2} \left(\phi + \frac{1}{qv} \dot{\eta} \right)^2 \right) + \frac{1}{4} f_{ab} f^{ab} - \frac{1}{2} e_a e^a + \frac{1}{2} \mathcal{D}_a \rho \mathcal{D}^a \rho - \frac{1}{2} \lambda^{-2} \dot{\rho}^2 + \frac{1}{4} \alpha \left(\rho^2 - v^2 \right)^2 \right) . \tag{3.70}$$

The time derivative of the projected field a_b is as given in Eq. (5.22), i.e.

$$\dot{a}_b = -\lambda e_b + \mathcal{D}_b \phi \, . \tag{3.71}$$

We will denote the momenta conjugate to the fields $\phi\,, a_a\,, \rho$ and η as $\pi^\phi\,, \pi^a\,, \pi$ and π_η respectively. As in the Maxwell case, $\dot{\phi} = \pounds_\xi \phi$ does not appear in Eq. (3.70) and provides the primary constraint

$$\pi^\phi = \frac{\delta L}{\delta \dot{\phi}} = 0 \, . \tag{3.72}$$

The other momenta corresponding to the projected fields are given by

$$\begin{aligned}
\pi^a &= \frac{\delta L}{\delta \dot{a}_a} = -e^a\,, \\
\pi &= \frac{\delta L}{\delta \dot{\rho}} = \lambda^{-1}\dot{\rho}\,, \\
\pi_\eta &= \frac{\delta L}{\delta \dot{\eta}} = \frac{\lambda^{-1} q\rho^2}{v}\left(\phi + \frac{1}{qv}\dot{\eta}\right)\,.
\end{aligned} \tag{3.73}$$

The canonical Poisson brackets of the theory are

$$\begin{aligned}
\left[\phi(x), \pi^\phi(y)\right]_P &= \delta(x,y)\,, \\
\left[a_a(x), \pi^b(y)\right]_P &= \delta_a^b\delta(x,y)\,, \\
\left[\rho(x), \pi(y)\right]_P &= \delta(x,y)\,, \\
\left[\eta(x), \pi_\eta(y)\right]_P &= \delta(x,y)\,.
\end{aligned} \tag{3.74}$$

The canonical Hamiltonian H_C follows from the Legendre transform

$$\begin{aligned}
H_C &= \int\limits_\Sigma dV_x\,\left(\pi^b\dot{a}_b + \pi\dot{\rho} + \pi_\eta\dot{\eta}\right) - L \\
&= \int\limits_\Sigma dV_x\,\left(\lambda\left(\frac{1}{2}\pi^b\pi_b + \frac{1}{4}f_{ab}f^{ab} + \frac{1}{2}\frac{v^2}{\rho^2}\pi_\eta^2 + \frac{1}{2}\pi^2 + \frac{1}{2}\mathcal{D}_a\rho\mathcal{D}^a\rho + \frac{1}{4}\alpha\left(\rho^2 - v^2\right)^2\right.\right. \\
&\qquad \left.\left. + \frac{1}{2}q^2\rho^2 h^{ab}\left(a_a + \frac{1}{qv}\mathcal{D}_a\eta\right)\left(a_b + \frac{1}{qv}\mathcal{D}_b\eta\right)\right) + \pi^b\mathcal{D}_b\phi - qv\phi\pi_\eta\right)\,.
\end{aligned} \tag{3.75}$$

Using a Lagrange multiplier v_ϕ, we include the constraint in Eq. (3.72) to H_C to define

$$\tilde{H} = H_C + \int\limits_\Sigma dV_x\, v_\phi\pi^\phi\,. \tag{3.76}$$

3.3.1 The Dirac-Bergmann formalism

The Dirac-Bergmann formalism can be applied to Eq. (3.76) to determine all the constraints of the theory. We perform the consistency check on $\Omega_1 = \pi^\phi$ by evaluating the Poisson bracket between π^ϕ and the Hamiltonian $\tilde{H}$ by using a smearing function ϵ as follows

$$\begin{aligned}\int_\Sigma dV_y \epsilon(y)\dot{\pi}^\phi(y) &= \int_\Sigma dV_y \epsilon(y) \left[\pi^\phi(y), \tilde{H}\right]_P \\ &= \int_\Sigma dV_y \epsilon(y) \left[\pi^\phi(y), \int_\Sigma dV_x \pi^b(x) \mathcal{D}_b^x \phi(x) - qv\phi\pi_\eta\right]_P \\ &= -\oint_{\partial\Sigma} da_y\, \epsilon(y) n_b^y \pi^b(y) + \int_\Sigma dV_y\, \epsilon(y) \left(\mathcal{D}_b^y \pi^b(y) + qv\pi_\eta\right) . \end{aligned} \quad (3.77)$$

As in the Maxwell case, the smearing function is assumed to be regular at the horizons, which leads to the non-vanishing surface term in Eq. (3.77). Thus the consistency of $\dot{\Omega}_1 \approx 0$ requires the constraint

$$\Omega_2 = -n_b \pi^b \Big|_{\mathcal{H}} + \mathcal{D}_b \pi^b + qv\pi_\eta \approx 0\,. \quad (3.78)$$

Any calculations involving the constraint in Eq. (3.78) requires that we smear and integrate it over the volume. Using a smearing function ϵ which is regular at the horizon, we explicitly have

$$\int_\Sigma dV_x\ \epsilon(x)\Omega_2(x) = -\oint_{\partial\Sigma} da_x\ n_b \pi^b(x) + \int_\Sigma dV_x\ \mathcal{D}_b \pi^b + \int_\Sigma dV_x\ qv\pi_\eta \approx 0\,. \quad (3.79)$$

Thus the surface term in Eq. (3.78) is to be understood as providing surface integrals over the horizons of the background.

We can now include this constraint with its multiplier into the existing Hamiltonian given in Eq. (3.76) to define

$$H_T = \tilde{H} + \int_\Sigma dV_x\ \left(v_1 \left(\mathcal{D}_b \pi^b + qv\pi_\eta\right)\right) - \oint_{\partial\Sigma} da_x\ v_1 n_b \pi^b\,. \quad (3.80)$$

It is straightforward to verify that, just as in the Maxwell case, $\dot{\Omega}_2 = [\Omega_2\,, H_T]_P = 0$. Thus there are no further constraints of the theory and the total Hamiltonian is given by Eq. (3.80). The multipliers v_1 and v_ϕ may be determined from the equations of motion for ϕ and a_b. The evolution of ϕ is given by

$$\int_\Sigma dV_y \epsilon(y)\dot{\phi}(y) = \int_\Sigma dV_y \epsilon(y)\left[\phi(y), H_T\right]_P$$
$$= \int_\Sigma dV_y \epsilon(y) v_\phi(y)\,. \tag{3.81}$$

Hence $v_\phi = \dot{\phi}$. Likewise, the evolution of a_b is given by

$$\int_\Sigma dV_y\, \epsilon(y)\dot{a}_b(y) = \int_\Sigma dV_y\, \left[\epsilon(y)a_b(y), H_T\right]_P$$
$$= \int_\Sigma dV_y \epsilon(y)\left[\lambda(y)\pi_b(y) + \mathcal{D}^y_b\phi(y) - \mathcal{D}^y_b v_1(y)\right]\,. \tag{3.82}$$

The term in the parenthesis in the last line of Eq. (3.82) agrees with $\pounds_\xi a_b$ provided $\mathcal{D}_b v_1 = 0$. Without any loss in generality, we can assume $v_1 = 0$. With this choice, Eq. (3.82) provides

$$\dot{a}_b = \lambda\pi_b + \mathcal{D}_b\phi\,. \tag{3.83}$$

With the multipliers determined, we thus have the following total Hamiltonian

$$H_T = \int_\Sigma dV_x \left(\lambda\left(\frac{1}{2}\pi^b\pi_b + \frac{1}{4}f_{ab}f^{ab} + \frac{1}{2}\frac{v^2}{\rho^2}\pi_\eta^2 + \frac{1}{2}\pi^2 + \frac{1}{2}\mathcal{D}_a\rho\mathcal{D}^a\rho + \frac{1}{4}\alpha\left(\rho^2 - v^2\right)^2\right.\right.$$
$$\left.\left. + \frac{1}{2}q^2\rho^2 h^{ab}\left(a_a + \frac{1}{qv}\mathcal{D}_a\eta\right)\left(a_b + \frac{1}{qv}\mathcal{D}_b\eta\right)\right) + \pi^b\mathcal{D}_b\phi - qv\phi\pi_\eta + \dot{\phi}\pi^\phi\right)\,. \tag{3.84}$$

3.3.2 Gauge transformations and Gauge fixing

To find the gauge transformations of the fields, we construct the generator as a general linear combination of the two first class constraints of the theory

$$\Delta(y) = \int_\Sigma dV_y \left(\alpha_1(y)\Omega_1(y) + \alpha_2(y)\Omega_2(y)\right)\,, \tag{3.85}$$

where $\Omega_1 = \pi^\phi$, Ω_2 as given in Eq. (3.78) and $\alpha_1\,,\alpha_2$ two arbitrary differentiable functions. The non-vanishing transformations on the fields are

$$\begin{aligned}
\delta\phi(x) &= [\phi(x)\,,\Delta(y)]_P = \alpha_1(x)\,,\\
\delta a_b(x) &= [a_b(x)\,,\Delta(y)]_P = -\mathcal{D}^x_b\alpha_2(x)\,,\\
\delta\eta(x) &= [\eta(x)\,,\Delta(y)]_P = qv\alpha_2(x)\,.
\end{aligned}\tag{3.86}$$

Identifying $\alpha_2(x) = -\alpha(x)$ and $\alpha_1(x) = \pounds_\xi\alpha(x)$ as in the case of gauge transformations of the Maxwell field, we recover the local gauge transformations under which the Lagrangian in Eq. (3.68) is invariant.

To gauge fix the theory, we will adopt the unitary gauge. Our treatment will closely follow that provided in [80] on flat spacetime. The complete set of constraints are now

$$\begin{aligned}
\Omega_1 &= \pi^\phi\,,\\
\Omega_2 &= \mathcal{D}_a\pi^a - n_a\pi^a\Big|_{\mathcal{H}} + qv\pi^\eta\,,\\
\Omega_3 &= \eta\,,\\
\Omega_4 &= \lambda\pi_\eta\frac{v^2}{\rho^2} - \phi qv\,.
\end{aligned}\tag{3.87}$$

While Ω_1 and Ω_2 are the two first class constraints which result from the theory, the additional constraints Ω_3 and Ω_4 are introduced to fix the gauge. The four constraints in Eq. (3.87) have the following non-vanishing Poisson brackets among themselves

$$\begin{aligned}
[\Omega_1(x)\,,\Omega_4(y)]_P = qv\delta(x,y) &= [\Omega_3(x)\,,\Omega_2(y)]_P\\
[\Omega_3(x)\,,\Omega_4(y)]_P &= \lambda(y)\frac{v^2}{\rho(y)^2}\delta(x,y)\,.
\end{aligned}\tag{3.88}$$

We note that the right hand side of the last equality in Eq. (3.88) can be expressed in either x or y, since the delta function is symmetric in its arguments. The Poisson brackets between these constraints define a matrix $C_{ab}\,(x,y) = [\Omega_a(x), \Omega_b(y)]_P$,

$$C(x,y) = \begin{pmatrix} 0 & 0 & 0 & qv \\ 0 & 0 & -qv & 0 \\ 0 & qv & 0 & \lambda(x)\frac{v^2}{\rho(x)^2} \\ -qv & 0 & -\lambda(x)\frac{v^2}{\rho(x)^2} & 0 \end{pmatrix} \delta(x,y)\,. \tag{3.89}$$

The inverse of this matrix is given by

$$C(x,y)^{-1} = \frac{1}{q^2v^2} \begin{pmatrix} 0 & \lambda(x)\frac{v^2}{\rho(x)^2} & 0 & -qv \\ -\lambda(x)\frac{v^2}{\rho(x)^2} & 0 & qv & 0 \\ 0 & -qv & 0 & 0 \\ qv & 0 & 0 & 0 \end{pmatrix} \delta(x,y)\,. \tag{3.90}$$

We can now define the Dirac brackets as in Eq. (2.21) for two dynamical entities A and B

$$[A\,,\,B]_D = [A\,,\,B]_P - \int dV_u \int dV_v \,[A\,,\,\Omega_\alpha(u)]_P\, C^{-1}_{\alpha\beta}\,(u,v)\,[\Omega_\beta(v)\,,\,B]_P\ . \tag{3.91}$$

Using Eq. (3.90) and Eq. (3.87), we find the following non-vanishing Dirac brackets

$$\left[a_a(x), \pi^b(y)\right]_D = \delta(x,y)\delta^b_a\,, \tag{3.92}$$

$$[\rho(x), \pi(y)]_D = \delta(x,y)\,, \tag{3.93}$$

$$[\phi(x), a_b(y)]_D = -\frac{\lambda(x)}{q^2\rho(x)^2}\mathcal{D}^y_a\delta(x,y)\,, \tag{3.94}$$

$$[a_b(x), \pi_\eta(y)]_D = \frac{1}{qv}\mathcal{D}^x_b\delta(x,y) \tag{3.95}$$

$$[\phi(x), \pi(y)]_D = 2\frac{v\lambda(y)}{q\rho(y)^3}\delta(x,y)\,. \tag{3.96}$$

Unlike the ordinary radiation gauge of the Maxwell field (on the Schwarzschild background), the Dirac brackets listed above have well defined limits at the horizon which are distinct from the Poisson brackets of the theory. Thus the unitary gauge is a good gauge on spherically symmetric backgrounds with horizons. With the above Dirac brackets, we can set the constraints in Eq. (3.87) to strongly vanish, i.e. $\Omega_i = 0\,; i = 1, 2, 3, 4$. The Dirac brackets in Eq. (3.92) and Eq. (3.93) are simply the canonical Poisson brackets. Thus $(a_b\,, \pi^b\,, \rho\,, \pi)$ are the independent fields of the reduced phase space. We also have $\pi^\phi = 0$, $\eta = 0$, $\Omega_2 = 0$ and $\Omega_4 = 0$, since the constraints now satisfy strong equalities throughout phase space. From Ω_2 of Eq. (3.87), we have

$$\pi_\eta(x) = \frac{1}{qv}\left(n^x_a \pi^a(x)\Big|_{\mathcal{H}} - \mathcal{D}^x_a \pi^a(x)\right) . \tag{3.97}$$

Eq. (3.97) along with Ω_4 of Eq. (3.87) further imply

$$\phi(x) = \frac{\lambda(x)}{q^2 \rho(x)^2} \mathcal{D}^x_a \pi^a(x)\,. \tag{3.98}$$

Eq. (3.97) and Eq. (3.98) may be viewed as the covariant expressions of those found in flat spacetime [80]. Due to the presence of surface terms, we see that the expressions for dependent variables of phase space are modified on backgrounds with horizons. Thus the modified Gauss law can have an effect on the physical charges of the theory. This will be further considered in the next subsection.

3.3.3 Charges

We will now consider the implication of the modified Gauss law constraint on the physical charges of the abelian Higgs model. On static spherically symmetric black hole backgrounds, solutions of the Higgs field can either have $\rho = 0$ or $\rho = \pm v$ at and outside the horizons of the black hole, corresponding to it being in the false vacuum or true vacuum respectively. In the case of asymptotically flat black hole backgrounds, ρ cannot vanish on the horizon and must thus take on the values $\rho = \pm v$. In this case the black hole does not carry any electric charge [92, 93] and there is a vanishing electric flux in the spacetime. A known exception occurs in the case of solutions on black hole backgrounds with a cosmological horizon. On these backgrounds, black holes can carry an electric charge while the Higgs field is in the false vacuum ($\rho = 0$) [94] and the solution is similar to that of a Reissner Nordström de Sitter black hole. While we will not consider solutions of the Higgs field on spherically symmetric backgrounds, we recall the above results in the literature to specify that we will consider the case where an electric flux exists in the region outside the black hole event horizon. We have already seen that in the Maxwell case, the flux over the horizon may vanish without it having to do so outside the horizon. We will now show that a similar result carries over in the case of the Abelian Higgs field.

The charge is defined by integrating Eq. (3.78) over regions of the hypersurface Σ

$$\int\limits_{\Sigma} dV_x \; \Omega_2(x) = Q \tag{3.99}$$

Let us first consider the integration over the entire hypersurface. We will consider the case where the spacetime has an inner black hole horizon and an outer cosmological horizon, whose corresponding surface integrals are over $\partial\Sigma_H$ and $\partial\Sigma_C$ respectively. In this case Eq. (3.99) gives

$$\begin{aligned}
Q = \int\limits_{\Sigma} dV_x \; \Omega_2(x) &= \int\limits_{\Sigma} dV_x \; \mathcal{D}_a\pi^a + \int\limits_{\Sigma} dV_x \; qv\pi_\eta - \boldsymbol{\oint\limits_{\partial\Sigma_C}} da_x \; n_a\pi^a + \boldsymbol{\oint\limits_{\partial\Sigma_H}} da_x \; n_a\pi^a \\
&= \oint\limits_{\partial\Sigma_C} da_x \; n_a\pi^a - \oint\limits_{\partial\Sigma_H} da_x \; n_a\pi^a + \int\limits_{\Sigma} dV_x \; qv\pi_\eta - \boldsymbol{\oint\limits_{\partial\Sigma_C}} da_x \; n_a\pi^a + \boldsymbol{\oint\limits_{\partial\Sigma_H}} da_x \; n_a\pi^a \\
&\Rightarrow \int\limits_{\Sigma} dV_x \; qv\pi_\eta = Q \,,
\end{aligned} \tag{3.100}$$

where $\boldsymbol{\oint}$ in the above equalities indicates that the surface integral originates from the surface term in the Gauss law constraint. This is the same notation introduced in our consideration of the charges of the Maxwell field in the previous subsection. Were the usual Gauss law constraint to hold, we would only have the first three terms given in the second line of Eq. (3.100). This would imply that the charge results from the difference in the flux across the two horizons. Due to the modified Gauss law constraint, we however find the last equality of Eq. (3.100), namely that the charge is given by the volume integral of the charge density $qv\pi_\eta$ over the entire hypersurface.

Let us now consider the integration over a bounded subregion of Σ, which we denote as Σ_B. The volume integral is carried out from r_H up to some radius r_B, where $r_C > r_B > r_H$, with r_C and r_H denoting the radial distance to the cosmological horizon and black hole horizon respectively. By integrating the modified Gauss law constraint of Eq. (3.78) in this case, we find

$$\begin{aligned} Q_B &= \int\limits_{\Sigma_B} dV_x\ \Omega_2(x) = \int\limits_{\Sigma_B} dV_x\ \mathcal{D}_a\pi^a + \int\limits_{\Sigma_B} dV_x\ qv\pi_\eta + \oint\limits_{\boldsymbol{\partial\Sigma_H}} da_x\ n_a\pi^a \\ &= \oint\limits_{\partial\Sigma_B} da_x\ n_a\pi^a - \oint\limits_{\partial\Sigma_H} da_x\ n_a\pi^a + \int\limits_{\Sigma_B} dV_x\ qv\pi_\eta + \oint\limits_{\boldsymbol{\partial\Sigma_H}} da_x\ n_a\pi^a \\ &\Rightarrow Q_B = \int\limits_{\Sigma_B} dV_x\ qv\pi_\eta + \oint\limits_{\partial\Sigma_B} da_x\ n_a\pi^a\,. \end{aligned} \tag{3.101}$$

In the first line of Eq. (3.101), we see that one of the surface terms in the constraint does not contribute, since $\partial\Sigma_B$ is not a horizon of the spacetime. This leads to the last line of Eq. (3.101). The charge Q_B does not vanish, since the flux $\oint\limits_{\partial\Sigma_B} da_x\ n_a\pi^a$ and the integral of the $qv\pi_\eta$ carried out to radius r_B can be finite.

By considering the limit of $\partial\Sigma_B \to \partial\Sigma_{\mathcal{H}}$, the volume integral vanishes and the surface integrals cancel out, which leads to

$$Q_H = 0\,, \tag{3.102}$$

at the horizon of the black hole. We thus find that the result found for the Maxwell field also holds for the Abelian Higgs field. In particular, the modified Gauss law allows for a vanishing flux across the horizons and a non-vanishing flux outside the horizon.

3.4 Discussion

In this chapter, we argued that horizons modify the constraints of gauge theories and could therefore affect the observed charges and dynamics. We then proceeded to demonstrate this through the examples of the Maxwell field and the Abelian Higgs model. In both cases, we found that the Gauss law constraint now involves surface contributions from the horizons of spherically symmetric backgrounds. It may appear that similar surface terms could result from spatial boundaries. As we have mentioned previously however, all fields, including gauge fields, are required to be continuous and satisfy certain regularity conditions on spatial boundaries which restrict their behaviour.

We can effectively consider two kinds of spatial boundaries; either one which is present within a given manifold, or one which constitutes the physical end of the manifold. If spatial boundaries exist within a manifold, then any surface term must exist on either side of the boundary and will thus cancel out. When the boundary corresponds to the physical end of the manifold, regularity of the fields require that they vanish there. These conditions can be ensured through the choice of smearing functions, which would in these cases have to satisfy Dirichlet, Neumann or Robin boundary conditions, depending on the spatial boundary being considered. The surface terms derived in the presence of the Killing horizons exist precisely because the horizon prevents the 'other' side from being observed. This is one of the properties which distinguishes spatial sections of Killing horizons from an ordinary spatial boundary. The only requirement we can impose is that gauge invariant scalars constructed from the gauge fields need to be finite at the horizon. Thus Killing horizons lead to a much richer set of possiblities than spatial boundaries, for field theories and in particular gauge theories.

One of the implications of the modified Gauss law constraint we considered were the corresponding conserved charges. In the Maxwell case, we saw that a non-vanishing charge and electric flux can exist outside the horizon of the black hole. However, the surface terms in the Gauss law imply that the charge and flux vanish on the surface of the horizon. This has an interesting consequence in the case of the Abelian Higgs field. If the Gauss law were to not contain the horizon corrections, then the absence of charged black holes would imply the absence of electric flux in the spacetime. We showed that the modified Gauss law allows for the charge to vanish on the horizon, while admitting a non-vanishing electric flux across any surface outside the horizon (and within the cosmological horizon, should it be present).

We also considered the consequences of the modified Gauss law constraint on gauge transformations and gauge fixing. We found that gauge transformations of the fields in all cases are not altered, which was due to the presence of the surface terms in the Gauss law constraint. We then gauge fixed the theory and derived the resulting Dirac brackets. For the Abelian Higgs model, we adopted the unitary gauge, where the phase of the Higgs field was fixed. In the case of the Maxwell field, we gauge fixed the theory using the radiation gauge. In these gauges, we derived the covariant generalization of the known Dirac brackets on flat spacetime. However, we showed that the Dirac brackets of the Maxwell field in the radiation gauge, which involves the time-independent Green function of the spacetime Laplacian operator, does not appropriately fix the gauge at the horizons. Specifically, in using the expression of the Green function on the Schwarzschild background, we find that its contribution to Dirac bracket vanishes at the horizon. Thus this Dirac bracket reduces to the Poisson bracket when any one of its arguments is considered at the horizon of a Schwarzschild black hole.

We thus also considered a radiation gauge for the Maxwell field which involves a surface term analogous to that involved in the modified Gauss law. We now find that the Dirac bracket involves the inverse spatial Laplacian of the background. Using the expression for this Green function on the Schwarzschild background, we find that its contribution in the Dirac bracket does not vanish at the horizon. Thus the Dirac brackets in this gauge are distinguished from the Poisson brackets even at the horizon of the Schwarzschild background.

The radiation gauge Dirac brackets of the Maxwell field depend on how fields are fixed at the horizon. In the case where the fields are not fixed at the horizon, we derive Dirac brackets which involve the Green function of spacetime Laplacian of the background. On the other hand, the fields are fixed at the horizon through the inclusion of a surface term in the radiation gauge. In this case, the Green function is such that its contribution in the Dirac bracket does not vanish at the horizon. This leads to the difference in the limits of the Dirac brackets at the horizon of the Schwarzschild black hole. A part of this result can be understood from the operators corresponding to the Green functions involved in the Dirac brackets. The difference in the action of the covariant Laplacian operator from that of the spatial Laplacian operator on a time-independent scalar field F satisfies the following identity

$$\nabla_a \nabla^a F - \mathcal{D}_a \mathcal{D}^a F = \left(\lambda^{-1} \mathcal{D}^a \lambda\right) D_a F \,. \tag{3.103}$$

In the $r \to r_{\mathcal{H}}$ limit, Eq. (3.103) becomes

$$\left(\nabla_a \nabla^a F - \mathcal{D}_a \mathcal{D}^a F\right)_{r=r_{\mathcal{H}}} = \kappa_{\mathcal{H}} \left(\partial_r F\right)_{r=r_{\mathcal{H}}} \,, \tag{3.104}$$

where $\kappa_{\mathcal{H}}$ is the surface gravity of the black hole. We see that these operators in general do not agree at the horizon and only do so through an appropriate choice of boundary conditions. For instance, a Neumann boundary condition would imply that the operators are identical at the horizon, while this would not be true for either Dirichlet or Robin boundary conditions. The eigenvalues as well as the Green functions of the operators are thus related to the choice of boundary conditions imposed on the fields.

The constraint in Eq. (3.28) can also be expected to affect the quantization of fields on backgrounds with horizons. The quantization of gauge fields can be effectively carried out within the Hamiltonian BRST formalism, where the first class constraints of the theory and the inclusion of additional ghost fields now describe the BRST charge operator. This operator identifies the physical states of the theory and provides the ghost and gauge fixing actions [79, 95]. The inclusion of surface terms in the BRST charge operator will thus affect the physical states and the BRST invariant action, just as the first class constraints did modify the physical charges and the gauge fixing of the theory. Some of these topics will be further considered in Chapter 5.

3.A Derivation of the Inverse Spatial Laplacian

In this appendix we will provide the derivation of the inverse spatial Laplacian given in Eq. (3.63) using the method of multipole expansion. This appendix will be based on the treatment and results of [84], where the Green function of Eq. (3.60) was derived for the Schwarzschild and pure de Sitter backgrounds. Not all backgrounds will provide a closed form expression in terms of elementary functions as in the Schwarzschild case. The method can however be used on any spherically symmetric, asymptotically flat background whose spacetime metric is given by

$$ds^2 = -\lambda(r)^2 dt^2 + \frac{1}{\lambda(r)^2} dr^2 + r^2 d\Omega^2 \,, \tag{3.A.1}$$

where $\lambda(r) = 0$ at the horizons and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of the 2-sphere. In these coordinates, Eq. (3.60) can be expressed as

$$\sin\theta\partial_r \left(r^2\lambda(r)\partial_r\widetilde{G}\right) + \frac{1}{\lambda(r)}\partial_\theta\left(\sin\theta\partial_\theta\widetilde{G}\right) + \frac{1}{\lambda(r)\sin\theta}\partial^2_\phi\widetilde{G} = -4\pi\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')\,, \tag{3.A.2}$$

where the delta functions are now normalized according to

$$\int_{r_H}^{r_c} dr\delta(r-r') = 1\,, \quad \int_0^\pi d\theta\,\delta(\theta-\theta') = 1\,, \quad \int_0^{2\pi} d\phi\,\delta(\phi-\phi') = 1\,. \tag{3.A.3}$$

The $\delta(r-r')$ function is normalized in the region under consideration. While we have represented this as being from the event horizon r_H to the cosmological horizon r_c in Eq. (3.A.3), it should be interpreted according to the background being considered. For instance, in the case of the Schwarzschild background considered in Sec. [3.A.1] the integral ranges from r_H to ∞, while in the pure de Sitter case in Sec. [3.A.2] the integral is from 0 to r_C.

3.A.1 The Schwarzschild background

For the Schwarzschild background, $\lambda(r) = \sqrt{1-\frac{2m}{r}}$ and Eq. (3.A.2) can be expressed as

$$\sin\theta\partial_r\left(r^2\sqrt{1-\frac{2m}{r}}\partial_r\widetilde{G}\right) + \frac{1}{\sqrt{1-\frac{2m}{r}}}\partial_\theta\left(\sin\theta\partial_\theta\widetilde{G}\right) + \frac{1}{\sqrt{1-\frac{2m}{r}}\sin\theta}\partial^2_\phi\widetilde{G} = -4\pi\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')\,. \tag{3.A.1.1}$$

It will be convenient to make a change of variables from r to $y = \frac{r}{m} - 1$. After deriving the solution, we will change variables again to express the Green function in terms of the original coordinates. In terms of y, Eq. (3.A.1.1) takes the form

$$\sin\theta \left[\partial_y \left((y+1)^2 \sqrt{\frac{y-1}{y+1}} \partial_y \widetilde{G}\right) + \sqrt{\frac{y+1}{y-1}} \left(\frac{1}{\sin\theta}\partial_\theta \left(\sin\theta \partial_\theta \widetilde{G}\right) + \frac{1}{\sin^2\theta}\partial_\phi^2 \widetilde{G}\right)\right] = -4\pi \frac{\delta(y-y')}{m} \delta(\theta-\theta')\delta(\phi-\phi'), \tag{3.A.1.2}$$

with the point source now located at (y', θ', ϕ').

The angular delta functions satisfy the expressions in Eq. (3.A.3), while the y delta function now satisfies

$$\int_1^\infty dy\, \delta(y-y') = 1\,. \tag{3.A.1.3}$$

To find the solution of Eq. (3.A.1.2), we will first solve the corresponding homogeneous equation in the absence of the source

$$0 = \sqrt{\frac{y-1}{y+1}} \partial_y \left((y+1)^2 \sqrt{\frac{y-1}{y+1}} \partial_y \widetilde{G}\right) + \frac{1}{\sin\theta}\partial_\theta \left(\sin\theta \partial_\theta \widetilde{G}\right) + \frac{1}{\sin^2\theta}\partial_\phi^2 \widetilde{G}\,. \tag{3.A.1.4}$$

We can now use the spherical symmetry of the background to express the solution in terms of Legendre polynomials

$$\widetilde{G}(\vec{y}, \vec{y}') = \sum_{l=0}^{\infty} R_l(y, y') P_l(\cos\gamma)\,, \tag{3.A.1.5}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. We note that $P_l(\cos\gamma)$ is related to the spherical harmonics $Y_{l,m}(\theta,\phi)$ via the Legendre addition theorem (cf. Eqs. (14.30.8), (14.30.9), (14.30.11) of [96])

$$\frac{2l+1}{4\pi} P_l(\cos\gamma) = \sum_{m=-l}^{l} Y_{l,m}(\theta,\phi) Y^*_{l,m}(\theta',\phi')\,. \tag{3.A.1.6}$$

Eq. (3.A.1.6) implies that $P_l(\cos\gamma)$ further satisfies

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta P_l(\cos\gamma)\right)+\frac{1}{\sin^2\theta}\partial_\phi^2 P_l(\cos\gamma)=-l(l+1)P_l(\cos\gamma)\,, \tag{3.A.1.7}$$

$$\int_{-1}^{1}d\cos\theta\int_0^{2\pi}d\phi P_{l'}(\cos\gamma)P_l(\cos\gamma)=\delta_{ll'}\frac{4\pi}{2l+1}\,. \tag{3.A.1.8}$$

By substituting Eq. (3.A.1.5) in Eq. (3.A.1.4) and using Eq. (3.A.1.7) we get the differential equation

$$(1-y^2)\frac{d^2}{dy^2}R_l(y,y')-(2y-1)\frac{d}{dy}R_l(y,y')+l(l+1)R(y,y')=0\,. \tag{3.A.1.9}$$

We note that Eq. (3.A.1.9) is quite similar to the differential equation satisfied by Legendre functions

$$(1-y^2)\frac{d^2}{dy^2}P_\nu^\mu(y)-2y\frac{d}{dy}P_\nu^\mu(y)+\left[\nu(\nu+1)-\frac{\mu^2}{1-y^2}\right]P_\nu^\mu(y)=0\,. \tag{3.A.1.10}$$

It will therefore be useful to further adopt the ansatz $R_l(y,y')=B_l(y')P_\nu^\mu(y)A(y)$. By substituting this ansatz in Eq. (3.A.1.9) and making use of Eq. (3.A.1.10), we find the following equation

$$\begin{aligned}P_\nu^\mu(y)\left[(1-y^2)\frac{d^2}{dy^2}A(y)-(2y-1)\frac{d}{dy}A(y)-\left(\nu(\nu+1)-\frac{\mu^2}{1-y^2}-l(l+1)\right)A(y)\right]\\+\frac{d}{dy}P_\nu^\mu(y)\left[2(1-y^2)\frac{d}{dy}A(y)+A(y)\right]=0\,.\end{aligned} \tag{3.A.1.11}$$

This equation can only be satisfied if the coefficients of $\frac{d}{dy}P_\nu^\mu(y)$ and $P_\nu^\mu(y)$ separately vanish. If this was not the case, then Eq. (3.A.1.11) would violate the recurrence relations satisfied by the Legendre functions. The coefficients of $\frac{d}{dy}P_\nu^\mu(y)$ in Eq. (3.A.1.11) involves a differential equation for $A(y)$ which has the solution

$$A(y)=\left(\frac{y-1}{y+1}\right)^{\frac{1}{4}}\,. \tag{3.A.1.12}$$

Using Eq. (3.A.1.12) in Eq. (3.A.1.11), we find that the coefficient of $P_\nu^\mu(y)$ vanishes if $\mu=\frac{1}{2}$ and $\nu=l$. The other real independent solution of Eq. (3.A.1.9) can likewise be found by using the ansatz $R_l(y,y') = B_l(y')e^{-i\pi\mu}Q^\mu_\nu(y)A(y)$. Since $Q^\mu_\nu(y)$ also satisfies Eq. (3.A.1.10), we find the same solution for $A(y)\,,\mu$ and ν. Thus the general solution of Eq. (3.A.1.9) is given by

$$\begin{aligned} R_l(y,y') &= A_l(y')\left(\frac{y-1}{y+1}\right)^{\frac{1}{4}} P_l^{\frac{1}{2}}(y) + B_l(y')\left(\frac{y-1}{y+1}\right)^{\frac{1}{4}}\left(iQ_l^{\frac{1}{2}}(y)\right) \\ &= A_l(y')g_l(y) + B_l(y')f_l(y)\,, \end{aligned} \tag{3.A.1.13}$$

where the functions $g_l(y)$ and $f_l(y)$ involve Legendre functions of fractional degree, with the argument $y > 1$. These functions can be described in terms of hypergeometric functions, for which there exist several representations. A particular representation which we will use is (cf. pp 153-163, Table entry 10 and 28, of [97])

$$\begin{aligned} P^\mu_\nu(y) &= \frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{2^{\nu+1}\sqrt{\pi}\Gamma\left(-\nu-\mu\right)}\frac{y^{-\nu+\mu-1}}{(y^2-1)^{\frac{\mu}{2}}}\,{}_2F_1\left(\frac{1+\nu-\mu}{2},\frac{2+\nu-\mu}{2};\nu+\frac{3}{2};\frac{1}{y^2}\right) \\ &\quad+\frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(1+\nu-\mu\right)}\frac{y^{\nu+\mu}}{(y^2-1)^{\frac{\mu}{2}}}\,{}_2F_1\left(\frac{-\nu-\mu}{2},\frac{1-\nu-\mu}{2};-\nu+\frac{1}{2};\frac{1}{y^2}\right)\,, \\ e^{-i\pi\mu}Q^\mu_\nu(y) &= \frac{\sqrt{\pi}\Gamma\left(1+\nu+\mu\right)}{2^{\nu+1}\Gamma\left(\frac{3}{2}+\nu\right)}\frac{(y^2-1)^{\frac{\mu}{2}}}{y^{\nu+\mu+1}}\,{}_2F_1\left(\frac{\nu+\mu+2}{2},\frac{\nu+\mu+1}{2};\nu+\frac{3}{2};\frac{1}{y^2}\right)\,, \end{aligned} \tag{3.A.1.14}$$

With $\mu = \frac{1}{2}$ and $\nu = l$ we have the following expressions for $g_l(y)$ and $f_l(y)$

$$\begin{aligned} g_l(y) &= \frac{1}{\sqrt{y+1}}\left[\frac{1}{2^{l+1}}y^{-l-\frac{1}{2}}\,{}_2F_1\left(\frac{l+\frac{1}{2}}{2},\frac{l+\frac{3}{2}}{2};l+\frac{3}{2};\frac{1}{y^2}\right)\right. \\ &\qquad\left.+2^l y^{l+\frac{1}{2}}\,{}_2F_1\left(\frac{-l-\frac{1}{2}}{2},\frac{-l+\frac{1}{2}}{2};-l+\frac{1}{2};\frac{1}{y^2}\right)\right]\,, \\ f_l(y) &= \sqrt{y-1}\left[\frac{1}{2^l}y^{-l-\frac{3}{2}}\,{}_2F_1\left(\frac{l+\frac{5}{2}}{2},\frac{l+\frac{3}{2}}{2};l+\frac{3}{2};\frac{1}{y^2}\right)\right]\,. \end{aligned} \tag{3.A.1.15}$$

It turns out that the functions given in Eq. (3.A.1.15) admit expressions in terms of more elementary functions, which we will now describe. The hypergeometric functions contained in $g_l(y)$ in Eq. (3.A.1.15) have the following generic form and known representation

$$ {}_2F_1\left(a, a+\frac{1}{2}, 2a+1, \frac{1}{y^2}\right) = 2^{2a}\left(\frac{y+\sqrt{y^2-1}}{y}\right)^{-2a} . \tag{3.A.1.16}$$

$a = \frac{l+\frac{1}{2}}{2}$ and $a = \frac{-l-\frac{1}{2}}{2}$ provide the hypergeometric functions involved in the $g_l(y)$ expression. We thus find the following expression for $g_l(y)$

$$ g_l(y) = \frac{1}{\sqrt{2}\sqrt{y+1}}\left[\left(y+\sqrt{y^2-1}\right)^{-l-\frac{1}{2}} + \left(y+\sqrt{y^2-1}\right)^{l+\frac{1}{2}}\right]. \tag{3.A.1.17}$$

Likewise, the hypergeometric function given in $f_l(y)$ has the following form and representation in terms of elementary functions

$$ {}_2F_1\left(b, b+\frac{1}{2}, 2b, \frac{1}{y^2}\right) = \frac{2^{2b-1}y^{2b}}{\sqrt{y^2-1}}\left(y+\sqrt{y^2-1}\right)^{-2b+1} , \tag{3.A.1.18}$$

where $b = \frac{l+\frac{3}{2}}{2}$. We can hence write $f_l(y)$ as

$$ f_l(y) = \sqrt{2}\,\frac{\left(y+\sqrt{y^2-1}\right)^{-l-\frac{1}{2}}}{\sqrt{y+1}} . \tag{3.A.1.19}$$

The derivation of the Green function will require the Wronskian of the solutions in Eq. (3.A.1.15). Using the above expressions, we find that the Wronskian $W(g_l(y), f_l(y), y) = g_l(y)\partial_y f_l(y) - f_l(y)\partial_y g_l(y)$ is given by

$$ W(g_l(y), f_l(y), y) = -\frac{(2l+1)}{(1+y)^{\frac{3}{2}}\sqrt{y-1}} . \tag{3.A.1.20}$$

To describe the general form of the solution $\widetilde{G}(\vec{y}, \vec{y}')$ in the presence of the point source, we also need to consider the limits of the solutions given in Eq. (3.A.1.17) and Eq. (3.A.1.19) and their derivatives. The asymptotic limits of the argument are $y \to 1$ and $y \to \infty$, which correspond to $r \to 2m$ and $r \to \infty$ respectively. We note that $g_0(y)$ is a special case in that it is a constant, $g_0(y) = 1$, for all values of y.

For all the other terms we find the following. As $y \to 1$, both $g_l(y) \to 1$ and $f_l(y) \to 1$ for all values of l, i.e. they are both finite. However, all derivatives of $f_l(y)$ diverge as $y \to 1$, while $\frac{d}{dy} g_l(y) \to l(l+1)$ as $y \to 1$. Thus the near horizon solution must only contain $g_l(y)$, and we must set $B_l(y') = 0$ in Eq. (3.A.1.13) in the region between (y', θ', ϕ') and the event horizon of the black hole.

As $y \to \infty$ we find that $f_l(y) \to 0$ for all values of l and the derivatives of $f_l(y)$ are also well behaved. On the other hand, $g_l(y)$ diverges for $l \neq 0$. We must thus set $A_l(y') = 0$ in Eq.(3.A.1.13) to describe the region from (y', θ', ϕ') to ∞.

We can therefore write the solution in the two regions in the following way

$$\widetilde{G}\,(\vec{y}, \vec{y}') = \sum_{l=0}^{\infty} R_l(y, y') P_l(\cos\gamma) = \begin{cases} \displaystyle\sum_{l=0}^{\infty} A_l(y') g_l(y) P_l(\cos\gamma)\,, & (y < y') \\ \displaystyle\sum_{l=0}^{\infty} B_l(y') f_l(y) P_l(\cos\gamma)\,. & (y > y') \end{cases} \tag{3.A.1.21}$$

Continuity of $\widetilde{G}$ at $y = y'$ implies that $A_l(y') g_l(y') = B_l(y') f_l(y')$. Then we can define a constant C_l such that

$$C_l = \frac{A_l(y')}{f_l(y')} = \frac{B_l(y')}{g_l(y')}\,, \tag{3.A.1.22}$$

using which we can write the solution in the form

$$\widetilde{G}\,(\vec{y}, \vec{y}') = \sum_{l=0}^{\infty} R_l(y, y') P_l(\cos\gamma) = \begin{cases} \displaystyle\sum_{l=0}^{\infty} C_l f_l(y') g_l(y) P_l(\cos\gamma)\,, & (y < y') \\ \displaystyle\sum_{l=0}^{\infty} C_l g_l(y') f_l(y) P_l(\cos\gamma)\,, & (y > y') \end{cases} \tag{3.A.1.23}$$

where $f_l(y)$ and $g_l(y)$ are as given in Eq. (3.A.1.15). We can now determine the constants C_l by appropriately integrating Eq. (3.A.1.2). To begin with, we insert Eq. (3.A.1.23) into Eq. (3.A.1.2) and multiply both sides with $P_{l'}(\cos\gamma)$. By using

Eq. (3.A.1.8), the resulting integration with respect to θ and ϕ gives us

$$\frac{1}{2l+1}\left[\frac{d}{dy}\left((y+1)^2\sqrt{\frac{y-1}{y+1}}\frac{d}{dy}R_l(y)\right) - l(l+1)\sqrt{\frac{y+1}{y-1}}R_l(y)\right] = -\frac{\delta(y-y')}{m}\,. \tag{3.A.1.24}$$

Integrating Eq. (3.A.1.24) over an infinitesimal region from $y'-\epsilon$ to $y'+\epsilon$, we get

$$\begin{aligned} -\frac{1}{m} &= \frac{1}{2l+1}C_l(y'+1)^2\sqrt{\frac{y'-1}{y'+1}}\left[g_l(y')\left.\frac{df_l(y)}{dy}\right|_{y'+\epsilon} - f_l(y')\left.\frac{dg_l(y)}{dy}\right|_{y'-\epsilon}\right] \\ &= \frac{1}{2l+1}C_l(y'+1)^{\frac{3}{2}}\sqrt{y'-1}W(g_l(y'),f_l(y'),y') \\ &= -C_l\,, \end{aligned} \tag{3.A.1.25}$$

where in going from the second to the third equality in Eq. (3.A.1.25), we made use of the Wronskian given in Eq. (3.A.1.20). Thus we have determined that C_l is independent of l

$$C_l = \frac{1}{m}\,. \tag{3.A.1.26}$$

We can now write the solution of Eq. (3.A.1.2) as

$$\widetilde{G}\,(\vec{y}_<\,,\vec{y}_>) = \frac{1}{m}\sum_{l=0}^{\infty} g_l(y_<)\,f_l(y_>)P_l(\cos\gamma)\,, \tag{3.A.1.27}$$

where $y_< = \min(y\,,y')$ and $y_> = \max(y\,,y')$. Using Eq. (3.A.1.19) and Eq. (3.A.1.17), we find that the product $g_l(y_<)\,f_l(y_>)$ is given by

$$\begin{aligned} g_l(y_<)f_l(y_>) = \frac{1}{\sqrt{y_<+1}\sqrt{y_>+1}}&\left[\left(\frac{y_<+\sqrt{y_<^2-1}}{y_>+\sqrt{y_>^2-1}}\right)^{\frac{1}{2}+l}\right. \\ &\left.+\left(\left(y_<+\sqrt{y_<^2-1}\right)\left(y_>+\sqrt{y_>^2-1}\right)\right)^{-l-\frac{1}{2}}\right]. \end{aligned} \tag{3.A.1.28}$$

For the sake of notational convenience, let us define

$$A = y_> + \sqrt{y_>^2-1} \qquad \text{and} \qquad B = y_< + \sqrt{y_<^2-1}\,. \tag{3.A.1.29}$$

Using Eq. (3.A.1.28) and the standard expression for the generating function for Legendre polynomials

$$\sum_{l=0}^{\infty} t^l P_l(x) = \frac{1}{\sqrt{1-2xt+t^2}}\,, \tag{3.A.1.30}$$

we find that Eq.(3.A.1.27) takes the form

$$\widetilde{G}\,(\vec{y}_<\,,\vec{y}_>) = \frac{1}{m}\frac{1}{\sqrt{y_<+1}\sqrt{y_>+1}}\left[\frac{\sqrt{AB}}{\sqrt{A^2+B^2-2\,A\,B\,\cos\gamma}}+\frac{\sqrt{AB}}{\sqrt{A^2\,B^2+1-2\,A\,B\,\cos\gamma}}\right]. \tag{3.A.1.31}$$

To write the solution in terms of Schwarzschild coordinates, we simply make the substitution for y. We thus find

$$\begin{aligned}\widetilde{G}\,(\vec{r},\vec{r}') &= \\ &\frac{1}{\sqrt{rr'}}\left[\frac{\sqrt{(\kappa(r)r-m)\,(\kappa(r')r'-m)}}{\sqrt{(\kappa(r)r-m)^2+(\kappa(r')r'-m)^2-2\,(\kappa(r)r-m)\,(\kappa(r')r'-m)\,\cos\gamma}}\right.\\ &\left.+\frac{m\sqrt{(\kappa(r)r-m)\,(\kappa(r')r'-m)}}{\sqrt{(\kappa(r)r-m)^2\,(\kappa(r')r'-m)^2+m^4-2\,m^2\,(\kappa(r)r-m)\,(\kappa(r')r'-m)\,\cos\gamma}}\right],\end{aligned} \tag{3.A.1.32}$$

where we have defined $\kappa(r)=1+\lambda(r)=1+\sqrt{1-\frac{2m}{r}}$, and $\kappa(r')$ similarly.

3.A.2 The static de Sitter background

The derivation of the inverse spatial Laplacian could also be of interest on cosmological backgrounds, such as de Sitter space with a positive cosmological constant Λ. In this case we have $\lambda(r)^2 = 1-\frac{r^2}{L^2}$, where $L=\sqrt{\frac{3}{\Lambda}}$. We make a change of coordinates and write $y = \frac{r}{L}$. For this choice, Eq. (3.A.2) takes the form

$$\sin\theta \left[\partial_y\left(y^2\sqrt{1-y^2}\partial_y\widetilde{G}\right) + \frac{1}{\sqrt{1-y^2}}\left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\widetilde{G}\right) + \frac{1}{\sin^2\theta}\partial_\phi^2\widetilde{G}\right)\right] = -4\pi\frac{\delta(y-y')}{L}\delta(\theta-\theta')\delta(\phi-\phi')\,. \tag{3.A.2.1}$$

The delta functions for the angular variables satisfy Eq. (3.A.3), while the y delta function satisfies

$$\int_0^1 dy\delta(y-y') = 1\,.$$

As in the Schwarzschild case, we use the spherical symmetry of the background to expand $\widetilde{G}$ in terms of Legendre polynomials

$$\widetilde{G}(\vec{y},\vec{y}') = \sum_{l=0}^{\infty} R_l(y,y')P_l(\cos\gamma)\,. \tag{3.A.2.2}$$

By substituting Eq. (3.A.2.2) in Eq. (3.A.2.1) without the delta function source and using Eq. (3.A.1.7), we get the equation

$$\sqrt{1-y^2}\frac{d}{dy}\left(y^2\sqrt{1-y^2}\frac{d}{dy}R_l(y,y')\right) - l(l+1)R_l(y,y') = 0\,. \tag{3.A.2.3}$$

To find the general solution in this case, it will be convenient to use the ansatz $R_l(y,y') = B_l(y')A(y)P_\mu^\nu(\sqrt{1-y^2})$. Using this ansatz in Eq. (3.A.2.3) and proceeding as described from Eqs. (3.A.1.10) - (3.A.1.13), we find the following general solution

$$R_l(y,y') = A_l'(y')(y)^{-\frac{1}{2}}P_{\frac{1}{2}}^{l+\frac{1}{2}}(\sqrt{1-y^2}) + B_l'(y')(y)^{-\frac{1}{2}}P_{\frac{1}{2}}^{-l-\frac{1}{2}}(\sqrt{1-y^2})\,. \tag{3.A.2.4}$$

The Legendre functions described in Eq. (3.A.2.4) are of fractional order and degree, defined in the region $[-1,+1]$. These functions can be described in terms of hypergeometric functions (cf. p.166 of [97]), for which we use the following representation

$$\Gamma(1-\mu)P_\nu^\mu(x) = 2^\mu(1-x^2)^{-\frac{\mu}{2}}\,{}_2F_1\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}, -\frac{\nu}{2}-\frac{\mu}{2}; 1-\mu; 1-x^2\right)\,. \tag{3.A.2.5}$$

By using Eq. (3.A.2.5) in Eq. (3.A.2.4) we find

$$R_l(y,y') = A_l(y')g_l(y) + B_l(y')f_l(y)\,, \tag{3.A.2.6}$$

where $g_l(y)$ and $f_l(y)$ are now given by

$$\begin{aligned} g_l(y) &= y^l\,{}_2F_1\left(\frac{l}{2},\frac{l}{2}+1;\frac{3}{2}+l;y^2\right) \\ f_l(y) &= \frac{1}{y^{l+1}}\,{}_2F_1\left(\frac{-l-1}{2},\frac{-l+1}{2};\frac{1}{2}-l;y^2\right)\,. \end{aligned} \tag{3.A.2.7}$$

We note that these are not the solutions as those given in Eq. (3.A.1.15). Here $A_l(y')$ and $B_l(y')$ are real coefficients and the solutions themselves are positive and real in the region between 0 and +1. The Wronskian of the two solutions given in Eq. (3.A.2.7) satisfies the following relation

$$W(g_l(y),f_l(y),y) = -\frac{2l+1}{y^2\sqrt{1-y^2}}\,. \tag{3.A.2.8}$$

To proceed, we will need to consider the limits of the solutions given in Eq. (3.A.2.7) and their derivatives as $y\to 0$ and $y\to 1$. As before, $g_0(y)=1$, which follows from ${}_2F_1\left(0,1;\frac{3}{2};y^2\right)=1$. We will thus consider the limits of the functions other than $g_0(y)$ in the following. Unlike Eq. (3.A.1.15) about the Schwarzschild background, we were unable to find expressions for $f_l(y)$ and $g_l(y)$ in terms of elementary functions for arbitrary l. We will thus consider the limits of these functions using known relations satisfied by hypergeometric functions.

As $y\to 0$, the hypergeometric functions are always given by ${}_2F_1\left(a,b,c,0\right)=1$. We also have the following derivative relation satisfied by the hypergeometric functions

$$\frac{d}{dx}\,{}_2F_1\left(a,b,c,x\right) = \frac{a\,b}{c}\,{}_2F_1\left(a+1,b+1,c+1,x\right)\,, \tag{3.A.2.9}$$

Thus the $y\to 0$ limit of the solutions in Eq. (3.A.2.7) are entirely determined from its explicit y dependence, and are not dependent on the hypergeometric functions involved. From Eq. (3.A.2.7), we see that $g_l(y)$ and its first derivative vanish, while $f_l(y)$ and its first derivative diverge for all values of l, as $y \to 0$. We must therefore set $B_l = 0$ in Eq. (3.A.2.6) to have regular solutions in the region $(y\,,\theta\,,\phi) < (y'\,,\theta'\,,\phi')$.

As $y \to 1$, we can make use of the following limit

$$ {}_2F_1\left(a, b, c, 1\right) = \frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\,, \qquad \Re\left(a+b-c\right) < 0\,;\ c \neq 0, -1, -2, \cdots \tag{3.A.2.10} $$

We note that the hypergeometric functions given in Eq. (3.A.2.7) are of the form ${}_2F_1\left(a, a+1; 2a+\frac{3}{2}; y^2\right)$, where $a = \frac{l}{2}$ and $a = \frac{-l-1}{2}$ provide the hypergeometric functions contained in $g_l(y)$ and $f_l(y)$ respectively. In both cases, the condition $\Re\left(a+b-c\right) < 0$ is satisfied and thus f_l and g_l are both regular as $y \to 1$.

We will now consider the limits of the derivatives of $f_l(y)$ and $g_l(y)$ as $y \to 1$. In the case of the $g_l(y)$ hypergeometric functions, we can use the following integral representation

$$ {}_2F_1\left(a, b, c, x\right) = \frac{\Gamma\left(c\right)}{\Gamma\left(b\right)\Gamma\left(c-b\right)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt\,, \quad \left(\Re(c) > \Re(b) > 0\right)\,. \tag{3.A.2.11} $$

While $c > b$ for the $g_l(y)$ hypergeometric functions, this is not the case for the $f_l(y)$ functions (which have $c < b$ for all $l \neq 0$ and $c = b$ for $l = 0$). For the function ${}_2F_1\left(a, a+1; 2a+\frac{3}{2}; y^2\right)$, with $a = \frac{l}{2}$, Eq. (3.A.2.11) becomes

$$ {}_2F_1\left(\frac{l}{2}, \frac{l}{2}+1, l+\frac{3}{2}, y^2\right) = \frac{\Gamma\left(l+\frac{3}{2}\right)}{\Gamma\left(\frac{l}{2}+1\right)\Gamma\left(\frac{l+1}{2}\right)}\int_0^1 \left(\frac{t}{1-ty^2}\right)^{\frac{l}{2}}(1-t)^{\frac{l-1}{2}}dt\,, \tag{3.A.2.12} $$

while the derivative of this function takes the form

$$ \partial_y\left({}_2F_1\left(\frac{l}{2}, \frac{l}{2}+1, l+\frac{3}{2}, y^2\right)\right) = \frac{\Gamma\left(l+\frac{3}{2}\right)ly}{\Gamma\left(\frac{l}{2}+1\right)\Gamma\left(\frac{l+1}{2}\right)}\int_0^1 \left(\frac{t}{1-ty^2}\right)^{\frac{l}{2}+1}(1-t)^{\frac{l-1}{2}}dt\,. \tag{3.A.2.13} $$

Eq. (3.A.2.13) diverges in the limit $y \to 1$ for $l > 0$. Since this implies that the derivatives of $g_l(y)$ diverge in this limit, we must set $A_l = 0$ for all $l \neq 0$.

The form of the $f_l(y)$ solutions can be determined from the contiguity relations satisfied by the hypergeometric functions. We will let $y^2 = z$, in terms of which the solutions being sought are $f_l(z) = z^{-\frac{l+1}{2}} F_l(z)$, with $F_l(z)$ defined by

$$ {}_2F_1\left(\frac{-l-1}{2}, \frac{-l+1}{2}; -l+\frac{1}{2}; z\right) = F_l(z)\,. \tag{3.A.2.14}$$

Two hypergeometric functions which are contiguous are related to one another through certain differentiation formulas. We will consider the following two identities (cf. Eq.s (15.5.4) and (15.5.9) of [96]).

$$\begin{aligned} {}_2F_1\left(a-n, b-n; c-n; z\right) &= \frac{\Gamma(c-n)}{\Gamma(c)}\frac{(1-z)^{c+n-a-b}}{z^{c-1-n}}\frac{d^n}{dz^n}\left[(1-z)^{a+b-c}\, z^{c-1}\, {}_2F_1\left(a,b;c;z\right)\right]\,, \\ {}_2F_1\left(a,b;c-n;z\right) &= \frac{\Gamma(c-n)}{\Gamma(c)} z^{1+n-c}\frac{d^n}{dz^n}\left[z^{c-1}\, {}_2F_1\left(a,b;c;z\right)\right]\,. \end{aligned} \tag{3.A.2.15}$$

The two equations in Eq. (3.A.2.15) further imply the following relation

$$\begin{aligned} &{}_2F_1\left(a-n,\ b-n; c-2n; z\right) \\ &\quad = \frac{\Gamma(c-2n)}{\Gamma(c)} z^{1+2n-c}\frac{d^n}{dz^n}\left[(1-z)^{c+n-a-b}\frac{d^n}{dz^n}\left[z^{c-1}(1-z)^{a+b-c}\, {}_2F_1\left(a,b;c;z\right)\right]\right]\,. \end{aligned} \tag{3.A.2.16}$$

Using Eq. (3.A.2.14) and for $n = 1$, we find from Eq. (3.A.2.16)

$$F_{l+2}(z) = \left(1 - \frac{(2l+1)(2l+2)z}{3+8l+4l^2}\right)F_l(z) - \frac{(8l+4)z-(8l+2)z^2}{3+8l+4l^2}F_l'(z) - \frac{4(z^3-z^2)}{3+8l+4l^2}F_l''(z)\,, \tag{3.A.2.17}$$

Thus every solution $f_l(z)$ can be determined recursively from the lowest order solutions of $F_l(z)$. All even solutions follow from the lowest order even solution, $F_0(z)$, while all odd solutions can be derived from $F_1(z)$. These solutions have known expressions [97] and are given by

$$F_0(z) = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; z\right) = \sqrt{1-z}\,, \tag{3.A.2.18}$$

$$F_1(z) = {}_2F_1\left(-1, 0; -\frac{1}{2}; z\right) = 1\,. \tag{3.A.2.19}$$

All even solutions have the form $F_l(z) = \sqrt{1-z}D_l(z)$. By substituting $F_l(z) = \sqrt{1-z}D_l(z)$ in Eq. (3.A.2.17), we find the following recursion relation for $D_l(z)$

$$D_{l+2}(z) = \left[\left(1 - \frac{2l(2l+1)z}{3+8l+4l^2}\right) D_l(z) - \frac{(8l+4)z-(8l-2)z^2}{3+8l+4l^2} D_l'(z) - \frac{4(z^3-z^2)}{3+8l+4l^2} D_l''(z)\right]\,. \tag{3.A.2.20}$$

If m represents positive integers, then from Eq. (3.A.2.20) it can be confirmed that D_{2m} is a polynomial of of order $m-1$ in z. For odd l, we can directly use Eq. (3.A.2.19) in Eq. (3.A.2.17) recursively to find that $F_{2m+1}(z)$ is a polynomial of order m in z. Specifically, the substitution of $F_1(z) = 1$ in Eq. (3.A.2.17) leads to $F_3(z) = 1 - \frac{4}{5}z$, using which we find from Eq. (3.A.2.17) that $F_5(z) = 1 - \frac{4}{3}z + \frac{8}{21}z^2$, and so on.

Using these properties, it now follows that the $f_l(y)$ solutions have the form

$$\begin{aligned} f_l(y) &= \sum_{n=0}^{\frac{l-1}{2}} \frac{c_n}{y^{2n+2}}\,, && (l\,\mathrm{odd})\,, \\ &= \frac{\sqrt{1-y^2}}{y}\,, && (l=0)\,, \\ &= \sqrt{1-y^2}\sum_{n=1}^{\frac{l}{2}} \frac{c_n}{y^{2n+1}}\,, && (l\,\mathrm{even}\,; l\neq 0)\,, \end{aligned} \tag{3.A.2.21}$$

where $c_{\frac{l-1}{2}} = 1$ for the odd l case, and $c_{\frac{l}{2}} = 1$ for the even l case. We see from Eq. (3.A.2.21) that the first derivatives of $f_l(y)$ differ for even and odd l. Specifically when $y \to 1$, the first derivative of $f_l(y)$ diverges when l is even and is finite when l

is odd. Regularity of the solutions requires that in the region $(y\,,\theta\,,\phi) > (y'\,,\theta'\,,\phi')$, we not only set $A_l = 0$ for all $l \neq 0$, but also set $B_l = 0$ for even l.

We can now determine the general solution $\widetilde{G}(\vec{y},\vec{y}')$ for the point source located at (y',θ',ϕ'). Away from the source, the solution is given by Eq.(3.A.2.2). As explained above, in the region $y < y'$ we simply set $B_l(y') = 0$ and sum over all l. In the region $y > y'$ we set $A_l(y') = 0$ for all $l \neq 0$ and sum over all odd l, but we in addition have the $g_0(y) = 1$ term which contributes a constant term. Thus, we can write

$$\widetilde{G}(\vec{y},\vec{y}') = \begin{cases} \displaystyle\sum_{l=0}^{\infty} A_l(y')g_l(y)P_l(\cos\gamma) & (y < y')\,, \\ \displaystyle A_0' + \sum_{l=0}^{\infty} B_{2l+1}(y')f_{2l+1}(y)P_{2l+1}(\cos\gamma) & (y > y')\,. \end{cases} \tag{3.A.2.22}$$

Finally, we need to match these solutions at $y = y'$. This sets $A_0 = A_0'$, and leads us to define the constant $C_{2l+1} = \frac{A_{2l+1}(y')}{f_{2l+1}(y')} = \frac{B_{2l+1}(y')}{g_{2l+1}(y')}$. We also find that A_k vanishes for even $k(\neq 0)$. Then we can write

$$\widetilde{G}(\vec{y},\vec{y}') = C_0 + R_{2l+1}(y,y')P_{2l+1}(\cos\gamma)\,, \tag{3.A.2.23}$$

where $C_0 \equiv A_0$ is the constant zero-mode contribution, and

$$R_{2l+1}(y,y') = \begin{cases} \displaystyle\sum_{l=0}^{\infty} C_{2l+1}g_{2l+1}(y)f_{2l+1}(y') & (y < y')\,, \\ \displaystyle\sum_{l=0}^{\infty} C_{2l+1}f_{2l+1}(y)g_{2l+1}(y') & (y > y')\,. \end{cases} \tag{3.A.2.24}$$

Multiplying both sides of Eq. (3.A.2.1) with $P_{2l'+1}(\cos\gamma)$ and integrating with respect to θ and ϕ, we get

$$-\frac{\delta(y-y')}{L} = \frac{1}{4l+3}\left[\frac{d}{dy}\left(y^2\sqrt{1-y^2}\frac{d}{dy}R_{2l+1}(y,y')\right) - \frac{(2l+1)(2l+3)}{\sqrt{1-y^2}}R_{2l+1}(y,y')\right]\,, \tag{3.A.2.25}$$

where we have used Eq. (3.A.1.8). We next integrate over y from $y' - \epsilon$ to $y' + \epsilon$, i.e. over an infinitesimal region about the point source, for which we find

$$\begin{aligned}
-\frac{1}{L} &= \frac{1}{4l+3} C_{2l+1} y'^2 \sqrt{1-y'^2} \left[g_{2l+1}(y') \left(\frac{d}{dy} f_{2l+1}(y) \right) \Big|_{y'+\epsilon} - f_{2l+1}(y') \left(\frac{d}{dy} g_{2l+1}(y) \right) \Big|_{y'-\epsilon} \right] \\
&= \frac{1}{4l+3} C_{2l+1} y'^2 \sqrt{1-y'^2}\, W(g_{2l+1}(y'), f_{2l+1}(y'), y') \\
&= -C_{2l+1} \,, \qquad (3.A.2.26)
\end{aligned}$$

where we have made use of Eq. (3.A.2.8) in going from the second to the third equality in Eq. (3.A.2.26). Using this constant, we can write the Green function in the de Sitter case as

$$\widetilde{G}\left(\vec{y}_<, \vec{y}_>\right) = \frac{1}{L} \sum_{l=0}^{\infty} g_{2l+1}(y_<) f_{2l+1}(y_>) P_{2l+1}(\cos\gamma) \,, \qquad (3.A.2.27)$$

where $y_< = \min(y, y')$ and $y_> = \max(y, y')$ as before. As mentioned previously, we have not been able to write this in a simpler form as in the Schwarzschild case. We can nonetheless substitute for y in Eq. (3.A.2.7). By writing $y_< = \frac{r_<}{L}$ and $y_> = \frac{r_>}{L}$ in Eq. (3.A.2.27), we find the following solution in terms of r

$$\begin{aligned}
\widetilde{G}\left(\vec{r}_<, \vec{r}_>\right) = \frac{1}{r_>^2} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>} \right)^{2l+1} {}_2F_1\left(l + \frac{1}{2}, l + \frac{3}{2}, 2l + \frac{5}{2}, \frac{3r_<^2}{\Lambda} \right) \times \\
\times\, {}_2F_1\left(-l-1, -l, -2l - \frac{1}{2}, \frac{3r_>^2}{\Lambda} \right) P_{2l+1}(\cos\gamma) \,.
\end{aligned}$$

(3.A.2.28)

4 Constrained dynamics on Kerr backgrounds

In this chapter, I will generalize the treatment of the previous chapter to stationary, axisymmetric spacetimes. These backgrounds are endowed with a timelike Killing vector field ξ^a and a spacelike Killing vector field ω^a. While these are commuting Killing vector fields, they are not orthogonal to one another. Thus we cannot construct spatial hypersurfaces orthogonal to the timelike Killing vector field as in the case of spherically symmetric backgrounds. We can however construct spatial hypersurfaces orthogonal to a vector field, which is a non-Killing combination of the two Killing vector fields outside the horizons and Killing at the horizons of the background. Following this foliation, spatial sections of the horizons of the spacetime are boundaries of the spatial hypersurfaces, as in the case of spherically symmetric backgrounds. We define time evolution as the Lie derivative with respect to the timelike Killing vector field of the background. This ensures that the background, and in particular the horizons, remains invariant under the time evolution of the fields. Thus the dynamical evolution of the projected fields do not require any modification of the usual Poisson brackets. The first section of this chapter describes the foliation of the background and the definition of time derivatives needed to perform the Hamiltonian analysis on these backgrounds.

To determine the role of horizons on the constraints, we perform the Dirac-Bergmann formalism on the Maxwell field. As in the previous chapter, we find surface contributions to the Gauss law from the horizons of the background. The two first-class constraints generate the usual gauge transformations of the theory, i.e. they are the same as those which leave the Lagrangian invariant. One of the main subtleties of the Hamiltonian framework on this background involves the consideration of spatial hypersurfaces which are not orthogonal to the timelike Killing vector. It is thus necessary to verify that our foliation of the Kerr background provides a Hamiltonian which generates time evolution along the timelike Killing vector field. By deriving Hamilton's equations for the Maxwell field, we find that the Hamiltonian is consistent with the projected Maxwell equations which result from the Lagrangian.

We finally consider the gauge fixing of the Maxwell field in two gauges. We first consider the radiation gauge, where the resulting Dirac bracket is shown to involve the time-independent and axially symmetric covariant Green function of the background. The other gauge fixing choice we consider is the axial gauge. In this gauge, we find a Dirac bracket which does not arise in flat spacetime due to the non-vanishing shift vector of Kerr spacetimes. The inverse matrix of the Poisson bracket of second-class constraints, needed to construct the Dirac brackets, involve functions which are not obviously single-valued on the Kerr background. One of the appendices of this chapter provides the derivation of these solutions on the asymptotically flat Kerr background in the Boyer-Lindquist coordinates. The solutions are the Kerr spacetime generalization of the known solutions in flat spacetime. The surface term in the Gauss law constraint also modifies the expression for the electromagnetic scalar potential in the axial gauge on Kerr backgrounds.

4.1 Foliation of Kerr backgrounds

The spacetimes we will consider in this chapter are stationary and axisymmetric, which may possess one or more horizons (as in the Kerr-de Sitter case). Such backgrounds admit two Killing vector fields: a timelike ξ^a and a spacelike ω^a, whose normalization will be taken to be

$$\xi_a \xi^a = -\lambda^2 \,,$$
$$\omega^a \omega_a = f^2 \,. \tag{4.1}$$

The orbits of ω^a are spacelike and closed, i.e. ω^a is periodic. The Killing vectors also mutually commute with one another

$$[\xi, \omega]^a = \xi^b \nabla_b \omega^a - \omega^b \nabla_b \xi^a = 0 \,. \tag{4.2}$$

Since ξ^a is not orthogonal to ω^a, we cannot construct spatial hypersurfaces which are orthogonal to the timelike Killing vector field of the background. We can however consider the following combination of the Killing vector fields

$$\chi^a = \xi^a + \alpha \omega^a \,, \tag{4.3}$$

where α is defined through the contraction of the Killing vectors

$$\alpha = -\frac{1}{f^2} \xi_a \omega^a \,. \tag{4.4}$$

We note that α in general is not a constant. It now follows that this vector is timelike in the region where $\lambda^2 + \alpha^2 f^2$ is positive, since

$$\chi_a \chi^a = -\beta^2 = -(\lambda^2 + \alpha^2 f^2) \,. \tag{4.5}$$

We further note that by construction χ^a is orthogonal to ω_a, i.e. $\chi^a \omega_a = 0$. Despite being a combination of Killing vectors, χ^a itself is not Killing since

$$\pounds_\chi g_{cd} = 2\omega_{(c} \nabla_{d)} \alpha \,. \tag{4.6}$$

We now further assume the stationary and axisymmetric spacetime to be such that it contains spacelike 2-planes spanned by $\{\mu^a\,,\nu^a\}$, which form integral submanifolds that are orthogonal to both ξ^a and ω^a [98,99]. We will specifically consider this class of stationary and axisymmetric spacetimes in this chapter, which will be referred to as Kerr spacetimes. For Kerr backgrounds, χ_a can be shown to satisfy the Frobenius condition. From the assumption of spacelike 2-planes orthogonal to both ξ^a and ω^a it follows that

$$\xi_{[a}\omega_b\nabla_c\omega_{d]} = 0\,, \tag{4.7}$$

$$\omega_{[a}\xi_b\nabla_c\xi_{d]} = 0\,. \tag{4.8}$$

Eq. (4.7) and Eq. (4.8) now provide the following condition

$$\omega_{[a}\chi_b\nabla_c\chi_{d]} = 0\,. \tag{4.9}$$

Contracting Eq. (4.9) with $\chi^b\omega^a$ leads to the following result

$$\nabla_{[c}\chi_{d]} = -\beta^{-2}\chi_{[c}\nabla_{d]}\beta^2\,, \tag{4.10}$$

Thus χ^a satisfies the Frobenius condition

$$\chi_{[a}\nabla_b\chi_{c]} = 0\,. \tag{4.11}$$

The time coordinate is along χ_a, which is orthogonal to spatial hypersurfaces we will denote by Σ. Thus Σ are 'equal time' hypersurfaces of the foliated background. Due to our assumption that the background has at least one horizon, the boundaries of these hypersurfaces include at least one surface where $\beta^2 = 0$. It is known that these surfaces correspond to the Killing horizons of the background [99]. More specifically, α is a constant and χ^a is a Killing vector field on the $\beta^2 = 0$ surfaces. Thus χ^a is timelike in the region outside the event horizon of asymptotically flat backgrounds, or in the general case of black hole de Sitter backgrounds, between the event horizon and the cosmological horizon.

We will denote the mutually orthogonal basis on the spatial hypersurface Σ by $\{f^{-1}\omega^a\,,\mu^a\,,\nu^a\}$. We also note that μ^a and ν^a are not uniquely determined in general. The projection operator on Σ therefore have the following equivalent expressions

$$h^a_b = \delta^a_b + \beta^{-2}\chi^a\chi_b\,, \tag{4.12}$$

$$= f^{-2}\omega^a\omega_b + \mu^a\mu_b + \nu^a\nu_b\,. \tag{4.13}$$

From Eq. (4.12), the determinant of the spacetime metric can be written as

$$\sqrt{-g} = \beta\sqrt{h}\,. \tag{4.14}$$

Spatial sections of the Killing horizons $\mathcal{H}$ are closed, axially symmetric surfaces, which are submanifolds of Σ. The induced metric on $\mathcal{H}$ is given by

$$\sigma_{ab} = h_{ab} - n_a n_b\,, \tag{4.15}$$

where n^a points into the region where χ_a is timelike and $n_a n^a = 1$. Since the horizon is axially symmetric and ω^a is tangent to the hypersurface Σ, it also follows that $n_a\omega^a = 0$.

With Eq. (4.12) we can project any spacetime tensor onto the spacelike hypersurface Σ. The projected covariant derivative on the hypersurface will be denoted by $\mathcal{D}_a = h^b_a\nabla_b$, which satisfies $\mathcal{D}_a h_{bc} = 0$. By definition, we also have the following projection

$$\mathcal{D}_a t^{c\ldots e}_{b\ldots d} = h^{a'}_a h^{b'}_b h^c_{c'}\cdots h^{e'}_e h^d_{d'}\nabla_{a'}T^{c'\ldots e'}_{b'\ldots d'}\,, \tag{4.16}$$

where $T^{c\ldots e}_{b\ldots d}$ represents a spacetime tensor, while $t^{c\ldots e}_{b\ldots d}$ denotes its projection on Σ. In the Hamiltonian formalism to be described in what follows, we will consider the time evolution vector to be along ξ^a. Unlike our treatment of spherically symmetric backgrounds, the time evolution vector is not along the time coordinate χ_a. With our choice, $\alpha\omega^a$ and β represent what are known as the shift and lapse of the time evolution vector ξ^a. It is the lapse function β which vanishes at the horizons.

Let Ψ_A , $A = 1, \cdots N$, denote some N spacetime fields on the background. The action functional for Ψ_A is given by the integral of the Lagrangian density $\mathcal{L}$ over the four volume

$$S[\Psi_A] = \int dV_4^x \; \mathcal{L}(\Psi_A(x), \nabla_a \Psi_A(x))\,, \tag{4.17}$$

where dV_4^x is the volume element of the spacetime, which from Eq. (4.14) can be expressed as $dV_4^x = \beta\, dt\, dV_x$. Denoting the spacetime volume element in the orthonormal basis by ϵ_{abcd} and the spatial volume element of the hypersurface by ${}^{(3)}\epsilon_{bcd}$, we also note that

$$\chi^a \epsilon_{abcd} = \beta\, {}^{(3)}\epsilon_{bcd} = \xi^a \epsilon_{abcd}\,. \tag{4.18}$$

Thus the projected volume element has the correct form even though time evolution takes place along ξ^a while it is χ^a which is orthogonal to Σ. Let us now project the fields Ψ^A and denote them by Φ^A. From Eq. (4.3), we can write the time derivatives of the fields Φ^A in the following manner

$$\dot{\Phi}_A := \pounds_\xi \Phi_A = \pounds_\chi \Phi_A - \pounds_{\alpha\omega} \Phi_A\,. \tag{4.19}$$

From a given action as in Eq. (4.17) we can determine the following projected action

$$\begin{aligned} S &= \int dt \int_\Sigma dV_x \; \beta(x) \mathcal{L}(\Phi_A(x), \mathcal{D}_a \Phi_A(x), \pounds_\xi \Phi_A(x)) \\ &= \int dt \; L(\Phi_A(x), \mathcal{D}_a \Phi_A(x), \pounds_\xi \Phi_A(x))\,. \end{aligned} \tag{4.20}$$

From Eq. (4.20), we can define the momenta Π^A canonically conjugate to Φ^A

$$\Pi^A = \frac{\delta L}{\delta \dot{\Phi}_A}\,. \tag{4.21}$$

As in Eq. (2.9), the functional derivative is an 'equal-time' derivative evaluated on the hypersurface Σ ,

$$\frac{\delta\Phi_A(\vec{x},t)}{\delta\Phi_B(\vec{y},t)} = \delta_A^B\,\delta(x,y) = \frac{\delta\dot{\Phi}_A(\vec{x},t)}{\delta\dot{\Phi}_B(\vec{y},t)}\,. \tag{4.22}$$

The three-dimensional covariant delta function on Σ in Eq. (4.22) was defined in Eq. (2.10) and satisfies

$$\int\limits_{\Sigma} dV_y \delta(x,y) f(\vec{y},t) = f(\vec{x},t)\,. \tag{4.23}$$

We will sometimes write $(\vec{x},t)$ as x , etc. as we have done in the above expressions.

The Hamiltonian formalism and the treatment of constrained field theories now proceeds exactly as described in Chapter 2 by using the above foliation. As in the case of spherically symmetric backgrounds with horizons, we expect that the boundaries of Σ, namely the spatial sections of the Killing horizons of the background, will modify the constraints with additional surface contributions. This will be illustrated in the next section through the example of the Maxwell field.

4.2 The Maxwell field

The action for the Maxwell field on curved backgrounds is given by

$$S_{EM} = \int dV_4 \left(-\tfrac{1}{4}F_{ab}F_{cd}g^{ac}g^{bd}\right)\,, \tag{4.24}$$

where dV_4 is the four dimensional volume form on the manifold $\Sigma\times\mathbb{R}$, and $F_{ab} = \partial_a A_b - \partial_b A_a$. By defining the projected fields $a_a = h_a^b A_b$, $\phi = \xi^a A_a$, $e_a = -\beta^{-1}\chi^c F_{ca}$, $f_{ab} = h_a^c h_b^d F_{cd}$ and using Eq. (4.14), we have the following projected action on Kerr backgrounds

$$S_{EM} = -\int dt \int\limits_{\Sigma} dV_x \frac{\beta}{4}\left(f_{ab}f^{ab} - 2e_a e^a\right)\,. \tag{4.25}$$

From Eq. (4.19), we have

$$\begin{aligned} \dot{A}_b \equiv \pounds_\xi A_b &= \pounds_\chi A_b - \alpha \pounds_\omega A_b - (A_a \omega^a)\, \nabla_b \alpha \\ &= \chi^a F_{ab} + \nabla_b (A_a \xi^a) - \alpha \omega^a F_{ab} \,, \end{aligned} \tag{4.26}$$

The projection of Eq. (4.26) on Σ thus provides the following expression for $\dot{a}_b$

$$\dot{a}_b = -\beta e_b + \mathcal{D}_b \phi + \alpha f_{ba} \omega^a \,. \tag{4.27}$$

Since $\dot{\phi}$ is absent in Eq. (4.25), its conjugate momentum is a primary constraint

$$\frac{\partial L_{EM}}{\partial \dot{\phi}} = \pi^\phi = 0 \,. \tag{4.28}$$

The momenta conjugate to the a_b are given by

$$\pi^b = \frac{\partial L_{EM}}{\partial \dot{a}_b} = -e^b. \tag{4.29}$$

The canonical Poisson brackets of the theory are

$$\begin{aligned} \left[\phi(x), \pi^\phi(y)\right]_P &= \delta(x,y) \\ \left[a_a(x), \pi^b(y)\right]_P &= \delta^b_a \delta(x,y) \,. \end{aligned} \tag{4.30}$$

The canonical Hamiltonian follows from the usual definition

$$\begin{aligned} H_C &= \int\limits_\Sigma dV_x \, \left(\pi^b \dot{a}_b\right) - L \\ &= \int\limits_\Sigma dV_x \, \left(\beta \left(\frac{1}{2} \pi^b \pi_b + \frac{1}{4} f_{ab} f^{ab} \right) + \pi^b \mathcal{D}_b \phi + \alpha \pi^b f_{ba} \omega^a \right) \,. \end{aligned} \tag{4.31}$$

The Hamiltonian comprises of the usual energy density along with an energy current $\alpha \pi^b f_{ba} \omega^a$ due to the non-vanishing shift vector of the background. This current has been noted in other considerations of the Maxwell field on foliated backgrounds involving a non-vanishing shift vector [44,87]. With the aid of a Lagrange multiplier

v_ϕ, we can include the constraint of Eq. (4.28) in Eq. (4.31) to define the following Hamiltonian

$$H_0 = \int\limits_\Sigma dV_x \left(\beta\left(\frac{1}{2}\pi^b\pi_b + \frac{1}{4}f_{ab}f^{ab}\right) + \pi^b\mathcal{D}_b\phi + \alpha\pi^b f_{ba}\omega^a + v_\phi\pi^\phi\right) . \tag{4.32}$$

4.2.1 The Dirac-Bergmann formalism

We will now determine all the constraints of the theory by implementing the Dirac-Bergmann formalism. We need to check that the existing constraint of the theory $\dot{\pi}^\phi \approx 0$, is obeyed at all times. As constraints are densities, they must be smeared with an appropriate function. By evaluating the Poisson bracket of the smeared constraint $\epsilon\pi^\phi$ with the Hamiltonian, with the smearing function ϵ assumed to be regular on the horizons, we find

$$\begin{aligned}
\int\limits_\Sigma dV_y \epsilon(y)\dot{\pi}^\phi(y) &= \int\limits_\Sigma dV_y \epsilon(y)\left[\pi^\phi(y), H_0\right]_P \\
&= \int\limits_\Sigma dV_y \epsilon(y)\left[\pi^\phi(y), \int\limits_\Sigma dV_x \pi^b(x)\mathcal{D}^x_b\phi(x)\right]_P \\
&= -\oint\limits_{\partial\Sigma} da_y\, \epsilon(y) n^y_b \pi^b(y) + \int\limits_\Sigma dV_y\, \epsilon(y)\left(\mathcal{D}^y_b\pi^b(y)\right) .
\end{aligned} \tag{4.33}$$

In deriving the above result we used the canonical Poisson brackets given in Eq. (4.31) and an integration by parts. The n_b involved in the surface integral over the horizons of the spacetime is the unit normal to the surface of the horizon satisfying $n_b n^b = 1$ and $n_b\omega^b = 0$. Further, in the case of black hole de Sitter backgrounds for example, the surface integral should be considered as a sum over both horizons, where n_b is outward pointing at the black hole horizon and inward pointing at the cosmological horizon. The Schwarz inequality tells us that the surface integrand of Eq. (4.33) is bounded by a finite quantity. More specifically, we have assumed that ϵ is regular and finite at the horizon, while the Schwarz inequality applied to the remaining terms in the surface integrand produces

$$\left|n_b\pi^b\right| \leq \sqrt{\left|n_bn^b\right|\,\left|\pi_b\pi^b\right|}\,. \tag{4.34}$$

Since $n_bn^b = 1$ and $\pi^b\pi_b$ is a gauge invariant scalar present in the stress energy tensor, we see that the surface integral in Eq. (4.33) does not vanish.

Thus all integrals in Eq. (4.33) are finite and provide the following constraint

$$\Omega_2 = -n_b\pi^b\Big|_{\mathcal{H}} + \mathcal{D}_b\pi^b \approx 0\,. \tag{4.35}$$

We note that Ω_2 is a distribution and must be smeared and integrated over the hypersurface. The notation $|_{\mathcal{H}}$ symbolizes that the first term of Eq. (4.35) must be integrated with respect to the area element at the horizons of the background. For the smeared constraint, we can explicitly write

$$\int\limits_{\Sigma} dV_y\,\epsilon(y)\Omega_2(y) = -\oint\limits_{\partial\Sigma} da_y\,\epsilon(y)n_b^y\pi^b(y) + \int\limits_{\Sigma} dV_y\,\epsilon(y)\mathcal{D}_b^y\pi^b(y)\,. \tag{4.36}$$

In other words, while the bulk term in Eq. (4.35) provides the usual Gauss law constraint for all points of Σ, the additional surface contribution of Eq. (4.35) must be considered for all points at the horizon ($\partial\Sigma$).

Evaluating the Poisson bracket of $\epsilon\Omega_2$ with the Hamiltonian reveals that $\dot{\Omega}_2 = 0$ and that there are no further constraints. Thus the unconstrained Hamiltonian is given by

$$H_T = \int\limits_{\Sigma} dV_x\left(\beta\left(\frac{1}{4}f_{ab}f^{ab} + \frac{1}{2}\pi_a\pi^a\right) + v_1\left(\mathcal{D}_b\pi^b\right) + \pi^b\mathcal{D}_b\phi + \alpha\pi^bf_{ba}\omega^a + v_\phi\pi^\phi\right) - \oint\limits_{\partial\Sigma} da_x\,v_1n_b\pi^b\,. \tag{4.37}$$

The multipliers v_1 and v_ϕ may be determined from the equations of motion of ϕ and a_a. The evolution of ϕ is given by

$$\int\limits_\Sigma dV_y \epsilon(y)\dot{\phi}(y) = \int\limits_\Sigma dV_y \epsilon(y)\left[\phi(y), H_T\right]_P = \int\limits_\Sigma dV_y \epsilon(y) v_\phi(y)\,, \tag{4.38}$$

which tells us that we can set $v_\phi = \dot{\phi}$. The evolution of a_b gives

$$\begin{aligned}\int\limits_\Sigma dV_y\, \epsilon(y)\dot{a}_b(y) &= \int\limits_\Sigma dV_y\, \left[\epsilon(y)a_b(y), H_T\right]_P \\ &= \int\limits_\Sigma dV_y \epsilon(y)\left(\beta(y)\pi_b(y) + \mathcal{D}^y_b\phi(y) + \alpha f_{ba}\omega^a - \mathcal{D}^y_b v_1(y)\right)\,. \end{aligned}\tag{4.39}$$

Comparing Eq. (4.39) with Eq. (4.27), we deduce that $\mathcal{D}_b v_1 = 0$. With this choice, Eq. (4.39) reduces to

$$\dot{a}_b = \beta\pi_b + \mathcal{D}_b\phi + \alpha f_{ba}\omega^a\,, \tag{4.40}$$

Hence the total Hamiltonian takes the form

$$H_T = \int\limits_\Sigma dV_x \left(\beta\left(\frac{1}{4}f_{ab}f^{ab} + \frac{1}{2}\pi_a\pi^a\right) + \pi^b\mathcal{D}_b\phi + \alpha\pi^b f_{ba}\omega^a + \dot{\phi}\pi^\phi\right)\,. \tag{4.41}$$

The two first class constraints also generate gauge transformations on the fields. By evaluating the Poisson bracket of the fields ϕ and a_b with the general linear combination of the constraints $\Delta(y) = \int\limits_\Sigma dV_y \epsilon_1(y)\pi^\phi(y) + \epsilon_2(y)\Omega_2(y)$, we find that

$$\begin{aligned}\delta\phi(x) &= \left[\phi(x), \Delta(y)\right]_P = \epsilon_1(x) \\ \delta a_b(x) &= \left[a_b(x), \Delta(y)\right]_P = -\mathcal{D}^x_b\epsilon_2(x)\,. \end{aligned}\tag{4.42}$$

The transformations in Eq. (4.42) are similar to those resulting on spherically symmetric backgrounds with horizons Eq. (3.36). The gauge transformations which leave the Lagrangian in Eq. (4.24) invariant are $\delta A_b = \nabla_b\epsilon$. By projecting this expression using Eq. (4.12) we have

$$\delta\left(\phi + \alpha\omega^a a_a\right) = \pounds_\chi\epsilon\,, \qquad \delta a_b = \mathcal{D}_b\epsilon\,. \tag{4.43}$$

Eq. (4.42) is equivalent to Eq. (4.43), provided we identify $\epsilon_1(x) = \pounds_\xi \epsilon(x)$ and $\epsilon_2(x) = -\epsilon(x)$. We also note that these gauge transformations are valid at the horizon, since $\phi(x)$ and $a_b(x)$ in Eq. (4.42) can be located at any point x on Σ, which includes $\partial\Sigma$. The existence of such transformations on $\partial\Sigma$ is related to gauge parameters being regular at the horizons, which has been reflected in our choice of smearing functions.

4.2.2 Maxwell equations

Since the Hamiltonian density is integrated over the hypersurface χ, it is not obvious that the Poisson bracket with the Hamiltonian generates time evolution along ξ. As a check, we will here demonstrate that the Hamiltonian does generate the correct time evolution and is also consistent with the projected Maxwell equations. Using Eq. (4.12) on $\nabla_a F^{ab} = 0$, one can find the following equations

$$\pounds_\chi e_b = -D_a(\beta f^{ab}) \tag{4.44}$$

$$\pounds_\chi f_{ab} = -2D_{[a}\beta e_{b]} \,. \tag{4.45}$$

In order for the formulation provided in Sec. (4.1) to be consistent, Hamilton's equations must reproduce Eq. (4.44) and Eq. (4.45). Using Eq. (4.41), we find the following expressions upon evaluating the Poisson brackets

$$\dot{\pi}^b = \left[\pi^b, H_T\right] = \mathcal{D}_a(\beta f^{ab}) + \mathcal{D}_a\left(\alpha(\pi^a\omega^b - \pi^b\omega^a)\right) - \alpha n_a \pi^a \omega^b\Big|_{\mathcal{H}} \,, \tag{4.46}$$

$$\dot{f}_{ab} = [f_{ab}, H_T] = 2\mathcal{D}_{[a}\beta\pi_{b]} + 2\mathcal{D}_{[a}\left(\alpha f_{b]c}\omega^c\right) \,. \tag{4.47}$$

To proceed, it will be useful to note that since $\chi^a\omega_a = 0$, $\omega^c\nabla_c = \omega^c\mathcal{D}_c$. Also from contracting Eq. (4.6), we see that $\pounds_\omega \alpha = 0$. Thus, $\pounds_{\alpha\omega}$ of any spatially projected quantity can be written entirely in terms of the spatially projected covariant derivative. Let us first consider $\pounds_{\alpha\omega} f_{ab}$

$$\begin{aligned}\pounds_{\alpha\omega} f_{ab} &= \alpha\omega^c \mathcal{D}_c f_{ab} + f_{ac}\mathcal{D}_b(\alpha\omega^c) + f_{cb}\mathcal{D}_a(\alpha\omega^c) \\ &= 2\alpha\omega^c \mathcal{D}_{[b} f_{a]c} + f_{ac}\mathcal{D}_b(\alpha\omega^c) + f_{cb}\mathcal{D}_a(\alpha\omega^c) \\ &= -2\mathcal{D}_{[a}\left(\alpha f_{b]c}\omega^c\right) \,, \end{aligned} \tag{4.48}$$

where we used the Bianchi identity $\mathcal{D}_{[c} f_{ab]} = 0$ (which follows from $\nabla_{[c} F_{ab]} = 0$) in going from the first equality to the second equality of Eq. (4.48). Likewise, we find for $\pounds_{\alpha\omega}\pi^b$

$$\begin{aligned}\pounds_{\alpha\omega}\pi^b &= \alpha\omega^c\mathcal{D}_c\pi^b - \pi^c\mathcal{D}_c\alpha\omega^b \\ &= \mathcal{D}_c\left(\alpha(\omega^c\pi^b - \pi^c\omega^b)\right) - \alpha\omega^b\mathcal{D}_c\pi^c \,. \end{aligned} \tag{4.49}$$

We used the property that ω^c is Killing in going from the first equality to the second equality of Eq. (4.49). Substituting Eq. (4.48) in Eq. (4.47) and Eq. (4.49) in Eq. (4.46), we find

$$\pounds_\chi \pi^b = \mathcal{D}_a(\beta f^{ab}) + \alpha\omega^b\left(\mathcal{D}_a\pi^a - n_a\pi^a\Big|_{\mathcal{H}}\right) \approx \mathcal{D}_a(\beta f^{ab}) \,, \tag{4.50}$$

$$\pounds_\chi f_{ab} = 2\mathcal{D}_{[a}\beta\pi_{b]} \,. \tag{4.51}$$

By further substituting Eq. (4.29) in the above expressions, we get the projected Maxwell equations given in Eq. (4.44) and Eq. (4.45). The hypersurface Σ of Kerr backgrounds should be contrasted with those considered for spherically symmetric backgrounds, where the time evolution vector is orthogonal to the hypersurface. In the spherically symmetric case where there was no shift vector, Eq. (4.50) would hold throughout phase space and not just on the constraint subspace. While this is an interesting point to consider, this is of no consequence from the standpoint of constrained Hamiltonian dynamics. Thus the foliation and time evolution as presented in Sec. (4.1) of this chapter are consistent with the covariant Maxwell equations.

4.2.3 Gauge fixing

We will now fix the gauge of this theory in two different ways, first by adopting the radiation gauge and then the axial gauge. In the radiation gauge, the full set of constraints are

$$\begin{aligned}
\Omega_1 &= \pi^\phi \\
\Omega_2 &= \mathcal{D}_a \pi^a - n_a \pi^a \Big|_{\mathcal{H}} \\
\Omega_3 &= \phi \\
\Omega_4 &= \mathcal{D}^b \left(\beta a_b\right) \, .
\end{aligned} \tag{4.52}$$

The first two are the gauge constraints of the theory given in Eq. (4.28) and Eq. (4.35), while Ω_3 and Ω_4 are the gauge-fixing functions. The constraints are now second-class and require the construction of Dirac brackets. The non-vanishing Poisson brackets of the constraints in Eq. (4.52) are

$$\begin{aligned}
\left[\Omega_1(x), \Omega_3(y)\right]_P &= -\delta(x,y) \, , \\
\left[\Omega_2(x), \Omega_4(y)\right]_P &= \mathcal{D}^y_a \big(\beta(y) \mathcal{D}^a_y \delta(x,y)\big) \, .
\end{aligned} \tag{4.53}$$

The first Poisson bracket is simply the canonical relation given in Eq. (4.31). Using two smearing functions γ and ϵ which are regular on the horizons of the background, the second Poisson bracket is calculated as follows

$$\begin{aligned}
&\left[\int dV_x \gamma(x) \Omega_2(x), \int dV_y \epsilon(y) \Omega_4(y)\right]_P \\
&\qquad = \left[\int dV_x \left(\mathcal{D}^x_a \gamma(x)\right) \pi^a(x) \, , \int dV_y \beta(y) \big(\mathcal{D}^b_y \epsilon(y)\big) \, a_b(y)\right]_P \\
&\qquad = -\int dV_y \beta(y) \left(\mathcal{D}^y_a \gamma(y)\right) \big(\mathcal{D}^a_y \epsilon(y)\big) \\
&\qquad = -\oint da_y \epsilon(y) n^a_y \beta(y) \left(\mathcal{D}^y_a \gamma(y)\right) + \int dV_y \epsilon(y) \mathcal{D}^a_y \left(\beta(y) \mathcal{D}^y_a \gamma(y)\right) .
\end{aligned} \tag{4.54}$$

By using Schwarz's inequality on the surface integrand, we find

$$\begin{aligned} |n^a \beta D_a(\gamma)|^2 &\leq |n^a n_a| \beta^2 |(D_a\gamma)(D^a\gamma)| \\ &= \beta^2 (D_a\gamma)(D^a\gamma) \ . \end{aligned} \tag{4.55}$$

Due to the presence of β^2, the surface integral vanishes and only the second term in the last line of Eq. (4.54) contributes. The Poisson bracket in Eq. (4.54) can thus be written as

$$\left[\int dV_x \gamma(x)\Omega_2(x), \int dV_y \epsilon(y)\Omega_4(y)\right]_P = \int dV_y \epsilon(y) \int dV_x \gamma(x) \left(\mathcal{D}^a_y \left(\beta(y)\mathcal{D}^y_a \delta(x,y)\right)\right) \ , \tag{4.56}$$

which is the result given in Eq. (4.53). The matrix of the Poisson brackets between these constraints have a non-vanishing determinant and is invertible. This matrix $C_{\alpha\beta}(x,y) = [\Omega_\alpha(x), \Omega_\beta(y)]_P$ is given by

$$C(x,y) = \begin{pmatrix} 0 & 0 & -\delta(x,y) & 0 \\ 0 & 0 & 0 & \mathcal{D}^y_a\left(\beta(y)\mathcal{D}^a_y\delta(x,y)\right) \\ \delta(x,y) & 0 & 0 & 0 \\ 0 & -\mathcal{D}^y_a\left(\beta(y)\mathcal{D}^a_y\delta(x,y)\right) & 0 & 0 \end{pmatrix} . \tag{4.57}$$

The Dirac brackets require the inverse of the matrix given in Eq. (4.57). This bracket was defined in Eq. (2.21) for any two dynamical entities A and B (which may be functions or functionals on phase space)

$$[A\,,\,B]_D = [A\,,\,B]_P - \int dV_u \int dV_v \, [A\,,\,\Omega_\alpha(u)]_P \, C^{-1}_{\alpha\beta}(u,v) \, [\Omega_\beta(v)\,,\,B]_P \ . \tag{4.58}$$

Thus we need to find the inverse of the operator $\mathcal{D}^y_a\left(\beta(y)\mathcal{D}^a_y\right)$. Let us formally write the inverse as $G(x,y)$, i.e.

$$\mathcal{D}^y_a\left(\beta(y)\mathcal{D}^a_y G(x,y)\right) = -\delta(x,y) \ . \tag{4.59}$$

By projecting the spacetime Laplacian operator, we can derive the following identity

$$\nabla_a \nabla^a G = \beta^{-1} \mathcal{D}_b \left(\beta \mathcal{D}^b G \right) - \beta^{-2} \ddot{G} - 2\alpha \beta^{-2} \pounds_\omega \dot{G} - \alpha^2 \beta^{-2} \pounds_\omega^2 G \,, \tag{4.60}$$

where for simplicity we have dropped the explicit dependence on the coordinates. We now observe that when G is independent of time and $\pounds_\omega G = 0$, we have $\nabla_a \nabla^a G = \beta^{-1} \mathcal{D}_b \left(\beta \mathcal{D}^b G \right)$. From Eq. (4.60) it follows that the time-independent and axisymmetric covariant Green's function which satisfies

$$\nabla_a^y \nabla_y^a G(x,y) = -\beta(y)^{-1} \delta(x,y) \,, \tag{4.61}$$

is equivalent to Eq. (4.59). Thus the inverse of the matrix in Eq. (4.57) $C^{-1}_{\alpha\beta}(x,y)$ is given by

$$C^{-1}(x,y) = \begin{pmatrix} 0 & 0 & \delta(x,y) & 0 \\ 0 & 0 & 0 & G\,(x,y) \\ -\delta(x,y) & 0 & 0 & 0 \\ 0 & -G\,(x,y) & 0 & 0 \end{pmatrix} . \tag{4.62}$$

We can substitute Eq. (4.62) in Eq. (4.58) to find the following Dirac bracket for the fields

$$\left[a_a(x), \pi^b(y) \right]_D = \delta(x,y) \delta_a^b - \mathcal{D}_a^x \left(\beta(y) \mathcal{D}_y^b G\,(x,y) \right) \,. \tag{4.63}$$

This Dirac bracket might have the same properties as the radiation gauge Dirac brackets considered on spherically symmetric backgrounds. However, unlike the Green function of the spacetime Laplacian operator on spherically symmetric backgrounds, there is no known Green function of this operator on Kerr backgrounds which is defined for all points at and outside the event horizon of the black hole. Thus we cannot consider the limit of this bracket on some particular axisymmetric spacetime when any one of its arguments is at the horizon.

Given that our background is axisymmetric, it will also be interesting to consider the axial gauge. Our consideration of the axial gauge will generalize the treatment provided in [81] about flat spacetime. We adopt the basis $\{f^{-2}\omega^a\,,\mu^a\,,\nu^a\}$ described in Sec. (4.1) and will consider Eq. (4.16) in the following equations. The 'z' direction of the axial gauge can be represented by either μ^a or ν^a in the $\mu-\nu$ plane. Choosing our direction to be along μ^a, we have the following set of constraints in this gauge

$$\begin{aligned}
\Omega_1 &= \pi^\phi \\
\Omega_2 &= \mathcal{D}_b\pi^b - n_b\pi^b\Big|_{\mathcal{H}} \\
\Omega_3 &= \mu^a a_a \\
\Omega_4 &= \mu^a\mathcal{D}_a\phi + \beta\mu^a\pi_a + \alpha\mu^a f_{ac}\omega^c\,.
\end{aligned} \tag{4.64}$$

We have chosen Ω_4 to be $\dot{\Omega}_3 = [\Omega_3\,, H_T]_P$ using Eq. (4.41). The constraints now lead to the following non-vanishing Poisson brackets

$$\begin{aligned}
[\Omega_1(x),\Omega_4(y)]_P &= -\mu^a(y)\mathcal{D}^y_a\delta(x,y) = [\Omega_4(x),\Omega_1(y)]_P\ , \\
[\Omega_2(x),\Omega_3(y)]_P &= \mu^a(y)\mathcal{D}^y_a\delta(x,y) = [\Omega_3(x),\Omega_2(y)]_P\ , \\
[\Omega_3(x),\Omega_4(y)]_P &= \beta(y)\delta(x,y)\,.
\end{aligned} \tag{4.65}$$

The bracket $[\Omega_2(x)\,,\Omega_4(y)]$ vanishes in the absence of torsion since

$$\begin{aligned}
&\left[\int dV_x\gamma(x)\Omega_2(x),\ \int dV_y\epsilon(y)\Omega_4(y)\right]_P \\
&\qquad = \int dV_x\mathcal{D}^x_a(\gamma(x))\mathcal{D}^x_b\left(\alpha(x)\epsilon(x)\left(\mu^b(x)\omega^a(x) - \mu^a(x)\omega^b(x)\right)\right) \\
&\qquad\quad - \oint da_x\mathcal{D}^x_a(\gamma(x))n^x_b\left(\alpha(x)\epsilon(x)\left(\mu^b(x)\omega^a(x)\right)\right) \\
&\qquad = -\int dV_x\alpha(x)\epsilon(x)\left(\mu^b(x)\omega^a(x) - \mu^a(x)\omega^b(x)\right)\mathcal{D}^x_b\mathcal{D}^x_a(\gamma(x)) \\
&\qquad = 0\,.
\end{aligned} \tag{4.66}$$

The Poisson brackets of (4.65) provide the following matrix

$$C(x,y) = \begin{pmatrix} 0 & 0 & 0 & -\mu^a(y)\mathcal{D}^y_a\delta(x,y) \\ 0 & 0 & \mu^a(y)\mathcal{D}^y_a\delta(x,y) & 0 \\ 0 & \mu^a(y)\mathcal{D}^y_a\delta(x,y) & 0 & \beta(y)\delta(x,y) \\ -\mu^a(y)\mathcal{D}^y_a\delta(x,y) & 0 & -\beta(y)\delta(x,y) & 0 \end{pmatrix}. \tag{4.67}$$

The inverse of this matrix is needed for the Dirac brackets, which we denote as

$$C^{-1}(x,y) = \begin{pmatrix} 0 & -p(x,y) & 0 & q(x,y) \\ p(x,y) & 0 & -q(x,y) & 0 \\ 0 & -q(x,y) & 0 & 0 \\ q(x,y) & 0 & 0 & 0 \end{pmatrix}, \tag{4.68}$$

where $p(x,y)$ and $q(x,y)$ are two arbitrary functions defined on the Kerr background. By evaluating $\int dV_z C(x,z)C^{-1}(z,y) = \delta(x,y)$, we find that these functions must satisfy

$$\mu^a(y)\mathcal{D}^y_a q(x,y) = -\delta(x,y) \tag{4.69}$$

$$\mu^a(y)\mathcal{D}^y_a p(x,y) = -\beta(y)q(x,y) \tag{4.70}$$

The expressions for p and q on the asymptotically flat Kerr background in Boyer-Lindquist coordinates has been derived in Appendix 4.B. While we have not derived these functions for more general Kerr backgrounds, such as Kerr-de Sitter for example, we believe that the techniques used in Appendix 4.B are sufficiently general to provide the solutions in such cases. Using the matrix of Eq. (4.68), we derive the following non-vanishing Dirac brackets for the fields

$$[\phi(x), a_b(y)]_D = \mu_b(y)\beta(y)q(x,y) + \mathcal{D}^y_b p(x,y)\,, \tag{4.71}$$

$$\begin{aligned}\left[\phi(x), \pi^b(y)\right]_D = \mathcal{D}^y_a \big(&\alpha(y)q(x,y)(\mu^a(y)\omega^b(y) - \mu^b(y)\omega^a(y))\big) \\ &- n^y_a\alpha(y)q(x,y)(\mu^a(y)\omega^b(y))\Big|_{\mathcal{H}}\,,\end{aligned} \tag{4.72}$$

$$[a_b(x), \pi^c(y)]_D = \delta^c_b\delta(x,y) + \mu^c(y)\mathcal{D}^y_b q(x,y)\,. \tag{4.73}$$

The Dirac bracket in Eq. (4.72) is not present in flat space results following the axial gauge. It exists on Kerr backgrounds due to the non-vanishing shift vector, while the surface terms are due to the presence of horizons. All of the above Dirac brackets are in fact distributions, which need to be integrated over Σ for both arguments, x and y. The surface term in the second bracket implies that the integral over y is over the horizon area, while the integral over x is a volume integral. To clarify this result, the derivation of Eq. (4.72) has been provided in Appendix 4.A. Eqs. (4.71) - (4.73) ensure that all brackets involving $\mu^a a_a$ and the other constraints of Eq. (4.64) identically vanish. Following the implementation of Dirac brackets, all of the constraints in Eq. (4.64) vanish throughout phase space. The Hamiltonian in Eq. (4.41) after imposing the constraints of Eq. (4.64) now becomes

$$\begin{aligned}H_T = \int\limits_{\Sigma} dV_x \Bigg(&\beta\left(\frac{1}{4}f_{ab}f^{ab} + \frac{1}{2}(f^{-2}\omega^a\omega_b + \nu^a\nu_b)\pi_a\pi^b\right) + (f^{-2}\omega^a\omega_b + \nu^a\nu_b)\pi^b\mathcal{D}_a\phi \\ &-\frac{1}{2}\beta\mu^a\pi_a\mu^b\pi_b + \alpha\nu_b\nu^c\pi^b f_{ca}\omega^a\Bigg)\,.\end{aligned} \tag{4.74}$$

The axial gauge on Kerr backgrounds can also be shown to modify the expression for the scalar potential ϕ. From the expressions of Ω_2 and Ω_4 of Eq. (4.64), we have

$$\begin{aligned}&\mathcal{D}_b(f^{-2}\omega^b\omega_a\pi^a + \nu^b\nu_a\pi^a) - \left(n_b\nu^b\nu_a\pi^a\right)|_{\mathcal{H}} \\ &\quad = \mathcal{D}_b\left(\beta^{-1}\mu^b\mu^a\left(\mathcal{D}^a\phi + \alpha f_{ac}\omega^c\right)\right) - \beta^{-1}n_b\mu^b\mu^a\left(\mathcal{D}_a\phi + \alpha f_{ac}\omega^c\right)|_{\mathcal{H}}\,.\end{aligned} \tag{4.75}$$

From Eq. (4.75) it follows that ϕ depends non-trivially on π^b at the horizon, in contrast with results either about flat spacetime or curved backgrounds without horizons.

4.3 Discussion

In this chapter, we considered the constrained dynamics of field theories on Kerr backgrounds and have argued that the constraints will involve contributions from the horizons of the background. This was explicitly demonstrated through the Maxwell field, where the Gauss law constraint was shown to involve horizon contributions. In the previous section, we also demonstrated that gauge fixing the theory can lead to additional implications on the fields and their dynamics due to the horizons of the background. As in the spherically symmetric case, the surface contributions are not generic to arbitrary boundaries. In considering spatial boundaries for example, the boundary is either located within the manifold or, as in the case of spatial infinity, constitutes the physical end on the manifold. Thus, either surface terms exist on both sides of a spatial boundary and cancel out, or the boundary represents the physical end of a manifold. In this case, the smearing functions must vanish to ensure the regularity of the fields. Such conditions on smearing functions do not hold on the horizons, which thus contribute non-vanishing surface terms in the constraints. Gauge fields can be completely arbitrary at the horizon, provided gauge invariant observables constructed from them remain finite.

As in the case of spherically symmetric backgrounds considered in the previous chapter, the modified Gauss law allows for a non-vanishing flux within Σ, which vanishes at the horizons $\partial\Sigma$. In the case of Kerr backgrounds, the momenta π^b

involve both the physical electric and magnetic fields

$$\begin{aligned}\pi^b = -e^b &= \beta^{-1}\chi_a F^{ab} \\ &= \beta^{-1}\xi_a F^{ab} + \alpha\omega_a F^{ab} \,. \end{aligned} \tag{4.76}$$

The integration of the Gauss law will provide an expression for the physical charges and currents. We will refer to these contributions as the conserved charge and denote it by Q in the following. We consider a subregion Σ_B of the hypersurface Σ, whose (spatial) outer boundary will be denoted as $\partial\Sigma_B$ and whose inner boundary is the black hole horizon, denoted as $\partial\Sigma_H$. Then the charge enclosed in this region is

$$\begin{aligned}Q_B &= \int\limits_{\Sigma_B} \Omega_2 = \oint\limits_{\partial\Sigma_B} n_b\pi^b - \oint\limits_{\partial\Sigma_H} n_b\pi^b + \boldsymbol{\oint}\limits_{\boldsymbol{\partial\Sigma_H}} n_b\pi^b \\ &= \oint\limits_{\partial\Sigma_B} n_b\pi^b \end{aligned} \tag{4.77}$$

In the first line of Eq. (4.77), $\boldsymbol{\oint}$ denotes the surface integral due to the surface term in the Gauss law constraint, following the notation introduced in the previous chapter. Thus an observer at any bulk point in Σ will observe a non-vanishing charge and current. Taking the limit of the outer boundary to the event horizon, we find that the expression for the charge vanishes

$$Q_H = \lim_{\partial\Sigma_B \to \partial\Sigma_H} Q_B = 0 \tag{4.78}$$

This results from a missing surface term contribution in Eq. (4.77) which has a non-vanishing limit at the horizon. We can also consider backgrounds with a cosmological horizon denoted by $\partial\Sigma_C$ and radius $r_C > r_H$. The integral of the Gauss law constraint now provides an expression for the total charge, which vanishes since

$$\begin{aligned}Q &= \int\limits_{\Sigma} \Omega_2 = \oint\limits_{\partial\Sigma_C} n_b\pi^b - \oint\limits_{\partial\Sigma_H} n_b\pi^b + \boldsymbol{\oint}\limits_{\boldsymbol{\partial\Sigma_H}} n_b\pi^b - \boldsymbol{\oint}\limits_{\boldsymbol{\partial\Sigma_C}} n_b\pi^b \\ &= 0 \end{aligned} \tag{4.79}$$

As in the case of spherically symmetric backgrounds, we see that while a non-vanishing flux exists in the bulk of Σ, it vanishes on the horizons $\partial\Sigma$.

Apart from the observed charges on the background, another consequence of the modified constraints involves the gauge fixing of the theory. In Sec. (4.2) we considered two gauges - the radiation gauge and the axial gauge. For the radiation gauge considered in Eq. (4.52), we chose the covariant generalization of the gauge adopted in flat space. Unsurprisingly, the Dirac brackets for a_a and π^b in Eq. (4.63) is the covariant generalization of the flat space result, involving the Green function of the spacetime Laplacian operator. Unlike the Schwarzschild background, the solution of this Green function is not known about Kerr backgrounds. We can however expect that the radiation gauge in the absence of surface terms involves the same inadequacy as was noted in the case of the usual radiation gauge about the Schwarzschild background. We could therefore always include an additional non-vanishing surface term $n^b a_b|_{\mathcal{H}}$ to the constraint $\Omega_4 \approx 0$ in Eq. (4.52). This gives

$$\Omega_4 = \mathcal{D}^b \left(\beta a_b\right) - n^b a_b|_{\mathcal{H}} \, , \tag{4.80}$$

We now find that the Poisson bracket $[\Omega_2(x), \Omega_4(y)]_P$ has the following expression

$$[\Omega_2(x), \Omega_4(y)]_P = \mathcal{D}^x_a \left(\beta(x)\mathcal{D}^a_x \delta(x,y)\right) - n^x_a \mathcal{D}^x_a \delta(x,y)|_{\mathcal{H}} \, . \tag{4.81}$$

The resulting Dirac bracket will now require the Green function for the operator involved in Eq. (4.81), which has a non-trivial surface contribution. This is similar to the Green function for the inverse spatial Laplacian considered in the case of spherically symmetric backgrounds, which did affect the Dirac brackets of the Maxwell field at the horizon as noted in Eq. (3.64). This result is not specific to the radiation gauge. We also noted that the Dirac bracket resulting from the axial gauge, given in Eq. (4.72), has a non-trivial surface term which affects its limit at the horizon. Likewise, the expression for the scalar potential in this gauge which follows from Eq. (4.75) also involves corrections from the horizons of the background. These results, while indicative of horizon effects on the dynamics and quantization of fields, are nevertheless gauge dependent. Covariant and gauge invariant implications of fixing fields at the horizon may be determined by working within the BRST formalism. This will be considered in the following chapter.

4.A Derivation of the Dirac bracket $\left[\phi(x), \pi^b(y)\right]_D$

The Dirac bracket provided in Eq. (4.72) involves a surface term, which we will now clarify by providing the steps in its derivation. From Eq. (4.58) we have

$$\left[\phi(x), \pi^b(y)\right]_D = -\int dV_u \int dV_v \left[\phi(x),\, \Omega_\alpha(u)\right]_P C^{-1}_{\alpha\beta}\,(u,v)\left[\Omega_\beta(v)\,, \pi^b(y)\right]_P\,. \tag{4.A.1}$$

where we made use of the fact that $\left[\phi(x), \pi^b(y)\right]_P = 0$. The expression in Eq. (4.A.1) simplifies to

$$\left[\phi(x), \pi^b(y)\right]_D = -\int dV_u \int dV_v \left[\phi(x),\, \Omega_1(u)\right]_P C^{-1}_{14}\,(u,v)\left[\Omega_4(v)\,, \pi^b(y)\right]_P\,. \tag{4.A.2}$$

The Dirac bracket requires the Poisson bracket $\left[\Omega_4(v)\,, \pi^b(y)\right]_P$. Its expression can be determined through the use of smearing functions γ and ϵ which are regular at the horizon. We find

$$\begin{aligned}
&\left[\int dV_x \gamma(x)\Omega_4(x)\,, \int dV_y \epsilon(y)\pi^b(y)\right]_P \\
&\quad = \oint da_x n^x_a \epsilon(x)\gamma(x)\alpha(x)\mu^a(x)\omega^b(x) - \int dV_x \epsilon(x)\mathcal{D}^x_a\left(\alpha(x)\gamma(x)\left(\mu^a(x)\omega^b(x) - \mu^b(x)\omega^a(x)\right)\right) \\
&\quad\; = \int dV_x \gamma(x)\left(\alpha(x)\left(\mu^a(x)\omega^b(x) - \mu^b(x)\omega^a(x)\right)\right)\mathcal{D}^x_a\epsilon(x) \\
&\qquad\quad \Rightarrow \left[\Omega_4(x)\,, \pi^b(y)\right] = \alpha(x)\left(\mu^a(x)\omega^b(x) - \mu^b(x)\omega^a(x)\right)\mathcal{D}^x_a\delta(x,y)\,.
\end{aligned} \tag{4.A.3}$$

Eqs. (4.64),(4.A.3) and (4.68) can now be used to find the expression

$$\begin{aligned}
\left[\phi(x), \pi^b(y)\right]_D &= -\int dV_u \delta(x,u) q(u,v) \int dV_v \alpha(v)(\mu^a(v)\omega^b(v) - \mu^b(v)\omega^a(v))\mathcal{D}^v_a \delta(v,y) \\
&= -\int dV_v\, q(x,v)\alpha(v)(\mu^a(v)\omega^b(v) - \mu^b(v)\omega^a(v))\mathcal{D}^v_a \delta(v,y) \\
&= -\oint da_v\, \delta(v,y) n^v_a q(x,v)\alpha(v)(\mu^a(v)\omega^b(v)) \\
&\quad + \int dV_v \delta(v,y)\mathcal{D}^v_a \left(q(x,v)\alpha(v)(\mu^a(v)\omega^b(v) - \mu^b(v)\omega^a(v))\right) .
\end{aligned} \tag{4.A.4}$$

Recalling that the brackets are in fact densities which need to be integrated over the hypersurface for both x and y, we can express the result of Eq. (4.A.4) as given in Eq. (4.72).

4.B Axial gauge functions in Boyer-Lindquist coordinates

We will now explicitly derive the function $q(x,y)$ following Eq. (4.69). The Maxwell field is assumed to be defined on the Kerr background, for which we will adopt the usual Boyer-Lindquist coordinates $(t\,, r\,, \theta\,, \phi)$

$$\begin{aligned}
ds^2_{BL} = &-\left(\frac{\Delta - a^2\sin^2\theta}{\rho^2}\right) dt^2 + \frac{2\,a\sin^2\theta}{\rho^2}\left(\Delta - r^2 - a^2\right) dtd\phi \\
&+ \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)^2 - \Delta a^2\sin^2\theta\right) d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2\,,
\end{aligned} \tag{4.B.1}$$

where

$$\Delta = r^2 - 2Mr + a^2\,, \qquad \rho^2 = r^2 + a^2\cos^2\theta\,, \tag{4.B.2}$$

with M being the mass of the black hole and a the angular momentum per unit mass. In these coordinates, r and θ span the integral 2-submanifolds of foliated

Kerr spacetimes, which are orthogonal to both ξ_a $((dt)_a)$ and ω_a $((d\phi)_a)$. In Sec. 4.2 we noted that the axial gauge analysis could be carried out for any of the basis vectors of this submanifold. Here we will demonstrate this by deriving the function $q(x, y)$ separately for the cases $\mu^a = (\partial_r)^a$ and $\mu^a = (\partial_\theta)^a$. From the inverse metric in Boyer-Lindquist coordinates, we have

$$(\partial_r)^a = \left(0, \frac{\sqrt{\Delta}}{\rho}, 0, 0\right) , \qquad (\partial_\theta)^a = \left(0, 0, \rho^{-1}, 0\right) . \tag{4.B.3}$$

Likewise, the metric components of Eq. (4.B.1) provide the following definitions

$$\begin{aligned} \lambda^2 &= -\frac{\Delta - a^2\sin^2\theta}{\rho^2} \\ \alpha f^2 &= -\frac{a\sin^2\theta}{\rho^2}\left(\Delta - r^2 - a^2\right) \\ f^2 &= \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)^2 - \Delta a^2\sin^2\theta\right) , \end{aligned} \tag{4.B.4}$$

as well as the following expressions for β and $\sqrt{h}$

$$\begin{aligned} \beta &= \sqrt{-(\lambda^2 + \alpha^2 f^2)} = \left(1 + \frac{4Mr\,(a^2+r^2)}{\Delta\,(a^2\cos(2\theta) + a^2 + 2r^2)}\right)^{-\frac{1}{2}} \\ \sqrt{h} &= \frac{f\rho^2}{\sqrt{\Delta}} . \end{aligned} \tag{4.B.5}$$

Since Eq. (4.69) involves a delta function source, it will be convenient to first re-express it in terms of a second-order differential equation. For the case where $\mu^a = (\partial_r)^a$, Eq. (4.69) can be explicitly rewritten as

$$\frac{\sqrt{\Delta(r',\theta')}}{\rho(r',\theta')}\partial_{r'}(\partial_{r'} l(\vec{r},\vec{r}')) = -\frac{1}{\sqrt{h(r',\theta')}}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') , \tag{4.B.6}$$

where we have chosen $q(\vec{r},\vec{r}') = \partial_{r'} l(\vec{r},\vec{r}')$ and have considered the source at a fixed point $\vec{r}$. We now assume the following ansatz. We now assume the following ansatz

$$l(\vec{r},\vec{r}') = l(r,\theta,r')\delta(\theta-\theta')\delta(\phi-\phi') , \tag{4.B.7}$$

which simplifies Eq. (4.B.6) to

$$\delta(\theta-\theta')\mu^r(r',\theta')\partial_{r'}(\partial_{r'}l(r,\theta,r')) = -\frac{1}{\sqrt{h(r',\theta')}}\delta(r-r')\delta(\theta-\theta')\,. \tag{4.B.8}$$

The solution for $l(r,\theta,r')$ follows by first considering the homogeneous equation $\partial_{r'}(\partial_{r'}R(r,\theta,r'))=0$, whose general solution is

$$R(r,\theta,r') = C_1(r,\theta) + C_2(r,\theta)r'\,. \tag{4.B.9}$$

Denoting the horizon radius as r_H, the solution $C_2(r,\theta)r'$ is found to be valid only in the region $r_H \leq r' < r$ (since it diverges in the region $r' > r$). The solution $C_1(r)$ is valid everywhere on Σ. By matching these solutions at the point $r = r'$, the general solution for $l(r,r')$ can then be written as

$$\begin{aligned} l(r,\theta,r') &= C(r,\theta)r' \qquad (r' < r) \\ &= C(r,\theta)r \qquad (r' > r)\,. \end{aligned} \tag{4.B.10}$$

Substituting this solution in Eq. (4.B.8), integrating θ' over its entire range and r' from $r-\epsilon$ to $r+\epsilon$, we find that $C(r,\theta)$ is given by

$$C(r,\theta) = \left(f(r,\theta)\rho(r,\theta)\right)^{-1}\,. \tag{4.B.11}$$

This leads to the general solution

$$\begin{aligned} l(\vec{r},\vec{r}') &= \delta(\theta-\theta')\delta(\phi-\phi')\left(f(r,\theta)\rho(r,\theta)\right)^{-1}r' \qquad (r' < r) \\ &= \delta(\theta-\theta')\delta(\phi-\phi')\left(f(r,\theta)\rho(r,\theta)\right)^{-1}r \qquad (r' > r)\,. \end{aligned} \tag{4.B.12}$$

Differentiating this solution with respect to r' gives

$$q(\vec{r},\vec{r}') = \frac{1}{f(r,\theta)\rho(r,\theta)}\Theta\left(r-r'\right)\delta(\theta-\theta')\delta(\phi-\phi')\,, \tag{4.B.13}$$

where $\Theta\left(r-r'\right)$ has the property that it is 1 when $r_H < r' < r$ and 0 elsewhere. We can also consider the case where $\mu^a = (\partial_\theta)^a$ in Eq. (4.69). In this case, Eq. (4.B.6) becomes

$$\mu^\theta(r',\theta')\partial_{\theta'}(\partial_{\theta'}l(\vec{r},\vec{r}')) = -\frac{1}{\sqrt{h(r',\theta')}}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')\,. \tag{4.B.14}$$

By using the ansatz

$$l(\vec{r},\vec{r}') = l(r,\theta,\theta')\delta(r-r')\delta(\phi-\phi') \tag{4.B.15}$$

and performing the analogous procedure described above, we find the following general solution for $q(\vec{r},\vec{r}')$

$$q(\vec{r},\vec{r}') = \frac{\sqrt{\Delta(r,\theta)}}{f(r,\theta)\rho(r,\theta)}\Theta\left(\theta-\theta'\right)\delta(r-r')\delta(\phi-\phi')\,, \tag{4.B.16}$$

where $\Theta\left(\theta-\theta'\right)$ is now just the ordinary Heaviside step function. Using the solutions given in Eq. (4.B.13) and Eq. (4.B.16), we can proceed to solve Eq. (4.70) when μ^a is either $(\partial_r)^a$ or $(\partial_\theta)^a$. We can alternatively rewrite Eq. (4.70) as

$$\mu^a(y)\mathcal{D}^y_a\left(\beta^{-1}(y)\mu^a(y)\mathcal{D}^y_a p(x,y)\right) = \delta(x,y) \tag{4.B.17}$$

and solve $p(x,y)$ using the procedure given above. The solutions for $p(x,y)$ about the Kerr background, when μ^a is either $(\partial_r)^a$ or $(\partial_\theta)^a$, are not as simple as those of $q(x,y)$ and involve elliptic integrals. In the case of $\mu^a = (\partial_\theta)^a$, the equation is

$$\frac{1}{\rho(r',\theta')}\partial_{\theta'}\left(\beta^{-1}(r',\theta')\frac{1}{\rho(r',\theta')}\partial_{\theta'}p(\vec{r},\vec{r}')\right) = \frac{1}{\sqrt{h(r',\theta')}}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')\,, \tag{4.B.18}$$

whose solution is given by

$$p(\vec{r},\vec{r}') = -\frac{\Delta(r,\theta)}{f(r,\theta)\rho(r,\theta)}\frac{F\left(\theta'\middle|\frac{a^2\Delta}{(a^2+r^2)^2}\right)}{2(a^2+r^2)}\delta(r-r')\delta(\phi-\phi')\,, \tag{4.B.19}$$

where $F\left(\theta' \middle| k^2\right)$ is known as the elliptic integral of the first kind. The solution of Eq. (4.70) when $\mu^a = (\partial_r)^a$ involves elliptic integrals with much more complicated arguments and we were unable to find a simple expression as in Eq. (4.B.19). However in all cases, the functions p and q are curved spacetime generalizations of the $p(x, y)$ and $q(x, y)$ known in flat spacetime without boundaries [81].

5 Hamiltonian BRST treatment of constrained field theories

In the previous chapters, we demonstrated that the constraints of field theories are modified through contributions from the horizons of the background. Horizons were shown to affect the charges and via gauge fixing, the Dirac brackets of the theory. While our results for the Dirac brackets indicate that dynamics are affected by fixing fields at the horizons, they nevertheless depend on the choice of gauge. As discussed at the end of Chapter 2 of this thesis, the Dirac-Bergmann formalism also does not provide an ideal setting to investigate quantum properties of certain theories, such as in the case of the Yang-Mills field for example. The quantization of constrained field theories is best addressed within the BRST formalism [79, 95, 100–104], which is manifestly unitary and gauge invariant. Our treatment in this chapter is based on the Hamiltonian BRST framework [79, 95], through which the modified constraints derived using the Dirac-Bergmann formalism can be appropriately considered. This framework will in particular allow us to determine the effect of horizons on physically observable fields and the renormalizability of gauge theories. In the next section, we will extend the Hamiltonian BRST formalism on flat spacetime to spherically symmetric backgrounds with horizons. The definitions we adopt will provide the usual covariant action in the absence of any surface terms.

As examples, we will consider the Yang-Mills field and scalar electrodynamics. In both examples, we first derive the constraints and find the presence of horizon contributions to the Gauss law. We then extend the phase space to include the ghosts and their conjugate momenta. The BRST invariant path integral follows from the definitions of the BRST charge operator and the gauge fixing function. We find that the BRST transformations of the fields do not involve corrections from the horizons of the background. We then consider a gauge fixing function which corresponds to the radiation gauge with a horizon contribution and use it to derive the BRST invariant action. This gauge fixing choice leads to surface integrals at the horizons in the ghost and gauge fixing effective actions of the theory. By using the Zinn-Justin equation, we demonstrate that the renormalizability of all gauge theories of the Yang-Mills type are not affected by the presence of horizons.

The horizons of the background could also have interesting consequences on the physical states of the theory. We explore this aspect in the scalar electrodynamics example through the construction of a co-BRST charge. This charge is conserved, nilpotent and allows for a resolution of the (physical) singlet states of the Hilbert space. The co-BRST charge also provide expressions for dressed charges, which have been used to describe the static charges of scalar electrodynamics on flat spacetime. The presence of surface terms from the horizons in the gauge fixing function will be shown to modify this dressing and hence the description of the static charges of the theory, especially near the horizons.

We conclude this chapter with expressions of the globally Lorentz covariant effective actions of the Yang-Mills field and scalar electrodynamics, corresponding to the derived BRST invariant projected actions of the previous sections.

5.1 Hamiltonian BRST formalism

We will now briefly review the Hamiltonian BRST formalism for constrained field theories based on the treatment in [79, 95]. To proceed, we will first recall some material from Chapters 2 and 3 of this thesis.

5.1.1 Foliation of spherically symmetric backgrounds

The constrained field theories we will consider are defined on fixed spherically symmetric backgrounds with one or more horizons. The timelike Killing vector field ξ^a of the background, normalized according to $\xi^a \xi_a = -\lambda^2$, allows us to foliate the spacetime into a one parameter family of spatial hypersurfaces Σ. The induced metric h_{ab} on Σ and the projection operator h_a^b are given by

$$h_{ab} = g_{ab} + \lambda^{-2} \xi_a \xi_b \,, \quad h^a_b = \delta^a_b + \lambda^{-2} \xi^a \xi_b \,. \tag{5.1}$$

We will denote the Killing horizons of the background as $\mathcal{H}$. The spatial sections of $\mathcal{H}$ are submanifolds of Σ, with induced metric σ_{ab} which can be written as

$$\sigma_{ab} = h_{ab} - n_a n_b \,, \tag{5.2}$$

where n^a is the unit spatial normal to the surface of the horizons of the background, which points in the direction of increasing time. In this chapter, we will also consider the null normals of $\mathcal{H}$ defined in spacetime. We recall that the projected metric $\tilde{\sigma}_{ab}$ of any null hypersurface of $\mathcal{M}$ can be written as

$$\tilde{\sigma}_{ab} = g_{ab} + l_a k_b + l_a k_b \,, \tag{5.3}$$

where l_a and k_a are two null normals of the null hypersurface which satisfy

$$l_a l^a = 0 = k_a k^a \quad , \quad l_a k^a = -1 \,. \tag{5.4}$$

On the spherically symmetric backgrounds we are considering, we can describe l_a and k_a in terms of the normalized timelike Killing vector field $\lambda^{-1}\xi_a$ and the unit spatial normal n_a to the spatial sections of the null hypersurfaces

$$l_a = \frac{1}{\sqrt{2}}\left(\lambda^{-1}\xi_a + n_a\right) , \qquad k_a = \frac{1}{\sqrt{2}}\left(\lambda^{-1}\xi_a - n_a\right) , \tag{5.5}$$

These expressions holds for all null hypersurfaces of the background, including $\mathcal{H}$. We can now use Eq. (5.5) to write n_a as

$$n_a = \frac{1}{\sqrt{2}}\left(l_a - k_a\right) \tag{5.6}$$

Using Eq. (5.5) in Eq. (5.3) and then projecting with h^a_b given in Eq. (5.1), we find the following projected null hypersurface metric σ_{ab} on Σ

$$\sigma_{ab} = h^c_a h^d_b \tilde{\sigma}_{cd} = h_{ab} - n_a n_b \, , \tag{5.7}$$

which agrees with Eq. (5.2). We will use Eq. (5.6) to provide covariant expressions of the surface integrals in the effective actions derived in this chapter.

5.1.2 Hamiltonian BRST formalism for constrained field theories

Time derivatives are defined as the Lie derivative with respect to ξ^a. Beginning with any action for a given field theory, we can define the Hamiltonian on Σ and apply the Dirac-Bergmann formalism to determine all the constraints of the theory. The fields can either have an even or odd Grassmann parity, which will be particularly important in this chapter. Hamiltonian dynamics are governed by the graded Poisson brackets defined in Eq. (2.12), which satisfy the identities and relations given in Eq. (2.13). We assume that any second-class constraints resulting from the Dirac-Bergmann formalism have been eliminated through the construction of Dirac brackets. In this case, the result of the Dirac-Bergmann formalism is the total Hamiltonian (involving the first-class constraints Ω_a and their multipliers v^a)

$$H_T = \int\limits_{\Sigma} dV_x \ \left(H_0 + v^a \Omega_a\right) \,, \tag{5.8}$$

with dynamics defined by graded Dirac brackets. In what follows, we will simply refer to the graded brackets following the Dirac-Bergmann formalism as Poisson brackets. We further restrict our attention to theories whose first-class constraints satisfy a Lie algebra

$$\begin{aligned} [\Omega_a, \Omega_b]_P &= C^c_{ab} \Omega_c \,, \\ [H_0, \Omega_b]_P &= V^a_b \Omega_a \,, \end{aligned} \tag{5.9}$$

where C^c_{ab} and V^a_b are constants. The Hamiltonian BRST formalism will be described in this section for theories so defined in phase space. We refer the interested reader to [95] for further information on the Hamiltonian BRST approach.

In the Hamiltonian BRST formalism, we first construct an extended phase space through the definition of additional fields and their momenta. The multipliers and their conjugate momenta are treated as canonical variables which satisfy

$$[v^a(x), \pi^v_b(y)]_P = \delta^a_b \delta(x, y) \,. \tag{5.10}$$

π^v_b are constraints of the theory in the extended phase space. Two additional fields and their momenta are introduced. The first pair comprise the ghosts $\mathcal{C}^a$ and their canonically conjugate momenta $\mathcal{P}_a$, which are equal to the number of first-class constraints Ω_a and of opposite Grassmann parity. The second pair are the antighosts $\bar{\mathcal{C}}_a$ and their canonically conjugate momenta $\bar{\mathcal{P}}^a$, which are equal in number and the opposite Grassmann parity of the constraints π^v_b. The ghosts and antighosts satisfy the following Poisson brackets

$$[\mathcal{P}_b(x), \mathcal{C}^a(y)]_P = -\delta^a_b \delta(x, y) = \left[\bar{\mathcal{P}}^a(x), \bar{\mathcal{C}}_b(y)\right]_P \,, \tag{5.11}$$

with all other brackets involving the $\mathcal{C}^a \,, \mathcal{P}_a \,, \bar{\mathcal{C}}_a$ and $\bar{\mathcal{P}}^a$ vanishing. The extended phase space has an additional structure corresponding to the ghost number. The ghost numbers of the fields in phase space are

$$\begin{aligned} \mathrm{gh}\left(\mathcal{C}^a\right) &= 1 = \mathrm{gh}\left(\bar{\mathcal{P}}^a\right) \,, \\ \mathrm{gh}\left(\mathcal{P}_a\right) &= -1 = \mathrm{gh}\left(\bar{\mathcal{C}}_a\right) \,, \end{aligned} \tag{5.12}$$

where gh() indicates the ghost number. The ghost number of all other canonical fields vanish. Given the extended phase space, we will define the Hamiltonian BRST charge Q_{BRST} on spherically symmetric backgrounds as

$$Q_{\mathrm{BRST}} = \int\limits_{\Sigma} dV_x \left(\mathcal{C}^a(x)\Omega_a(x) + \frac{1}{2}\mathcal{P}_a(x)C^a_{bc}\mathcal{C}^b(x)\mathcal{C}^c(x) - i\bar{\mathcal{P}}^a(x)\pi^v_a(x) \right) \,. \tag{5.13}$$

The BRST charge is Grassmann odd and has ghost number $\mathrm{gh}(Q_{\mathrm{BRST}}) = 1$. BRST transformations of the fields are generated by its Poisson bracket with Q_{BRST}. Given a functional of the fields F, we will denote its BRST transformation by sF

$$sF = [F \,, Q_{\mathrm{BRST}}]_P \,. \tag{5.14}$$

If F has ghost number γ^a and mass dimension d^a, then sF has ghost number $\gamma^a + 1$ and mass dimension $d^a + 1$. The BRST transformations generated by Q_{BRST} in Eq. (5.14) are similar to the gauge transformations in Eq. (2.24) generated by the first-class constraints Ω_a. The crucial difference with the first-class constraints comes from the closure of the BRST charge off-shell, i.e. throughout phase space

$$[Q_{\mathrm{BRST}} \,, Q_{\mathrm{BRST}}]_P = 0 \,. \tag{5.15}$$

Since the BRST charge is Grassmann odd, Eq. (5.15) is simply $Q^2_{\mathrm{BRST}} = 0$. Hence BRST transformations are nilpotent, i.e. $s^2F = 0$ for all F. From Eq. (5.15) and the Jacobi identity, we also have $[[F \,, Q_{\mathrm{BRST}}]_P \,, Q_{\mathrm{BRST}}]_P = 0$. Thus any BRST invariant quantity is known up to a Poisson bracket $[F\,, Q_{\mathrm{BRST}}]_P$. In particular, we can define the following effective BRST invariant Hamiltonian

$$H_{eff} = H_0 - [\Psi\,, Q_{\mathrm{BRST}}]_P \ , \tag{5.16}$$

where Ψ must have the same Grassmann parity of Q_{BRST} and opposite ghost number $\mathrm{gh}(\Psi) = -1$, but can otherwise be specified arbitrarily. Ψ is a gauge fixing function in the Hamiltonian BRST formalism. By using the Legendre transform with the Hamiltonian in Eq. (5.16), we can define the following BRST invariant effective action

$$S_{eff} = \int dt \int\limits_{\Sigma} dV_x \ \left(\Pi^A \dot{\Phi}_A + \pi^v_b \dot{v}_b + \mathcal{P}^a \dot{\mathcal{C}}_a + \bar{\mathcal{P}}^a \dot{\bar{\mathcal{C}}}_a - H_{eff}\right) \ . \tag{5.17}$$

We note that "effective action" here does not refer to any consideration of renormalizability. The "quantum effective action" will be clearly defined later on in this chapter. Since Ψ can be specified arbitrarily, physical processes are independent of the choice of gauge fixing. This further implies the Fradkin-Vilkovisky theorem [102, 103], where the following path integral over all the canonical variables of the extended phase space $\mu^A = \{\Phi_A\,, \Pi^A\,, \pi^v_a\,, v^a\,, \mathcal{C}^a\,, \mathcal{P}_a\,, \bar{\mathcal{C}}_a\,, \bar{\mathcal{P}}^a\}$

$$Z = \int \left[\mathcal{D}\mu^A\right] \exp\left(iS_{eff}\right) \ , \tag{5.18}$$

is independent of the choice of Ψ. In most cases, especially in the derivation of the covariant action, Ψ can be represented as

$$\Psi = \int\limits_{\Sigma} dV_x \left(i\bar{\mathcal{C}}_A(x)\chi^A(x) + \mathcal{P}_A(x) v^A(x)\right) \ , \tag{5.19}$$

where χ^A is a function which is independent of the ghosts. The covariant effective action can be derived by using Eq. (5.19) in Eq. (5.16) and Eq. (5.17), followed by an integration of the momenta of the theory in Eq. (5.18). This will be considered in the following sections for the Yang-Mills field and scalar electrodynamics.

5.2 The Yang-Mills field

5.2.1 Derivation of the Gauss law constraint

We will now consider the specific example of the Yang-Mills field on spherically symmetric spacetimes with horizon(s). This subsection will use the formalism described in Chapter 3 of this thesis. The action for the Yang-Mills field is given by

$$S_{YM} = \int dV_4 \left(-\tfrac{1}{4} F^A_{ab} F_{Acd} g^{ac} g^{bd}\right) , \tag{5.20}$$

where dV_4 is the four dimensional volume form on the manifold $\Sigma \times \mathbb{R}$ (with metric g_{ab}), which upon following Eq. (3.4) can be expressed as $dV_4 = \lambda\, dt\, dV_x$. The field strength of the Yang-Mills field $F^A_{ab} = \partial_a A^A_b - \partial_b A^A_a - g C^A_{BC} A^B_a A^C_b$, is defined through the structure functions C^A_{BC}. We are thus dealing with the general $SU(N)$ case in a basis Λ_A which satisfies $\mathrm{Tr}\,(\Lambda_A \Lambda_B) = \frac{1}{2}\delta_{AB}$ and $[\Lambda_A\,, \Lambda_B] = C^C_{AB}\Lambda_C$, where δ_{AB} is the Kronecker delta function.

Defining the projected fields $a^A_a = h^b_a A^A_b$, $\phi^A = \xi^a A^A_a$, $e^A_d = -\lambda^{-1}\xi^c F^A_{cd}$ and $f^A_{ab} = h^c_a h^d_b F^A_{cd}$, we can write the projected action as

$$S_{YM} = -\int dt \int_\Sigma dV_x \frac{\lambda}{4}\left(f^A_{ab} f^{ab}_A - 2 e^A_a e^a_A\right) . \tag{5.21}$$

The expression for $\dot{a}^A_b$ follows from the Lie derivative of a^A_a with respect to the timelike Killing vector field ξ^a

$$\dot{a}^A_b = \pounds_\xi a^A_b = -\lambda e^A_b + \mathcal{D}_b \phi^A + g C^A_{BC} \phi^B a^C_b . \tag{5.22}$$

From Eq. (5.21), we have the following momenta of the theory

$$\pi^b_A = \frac{\partial L_{EM}}{\partial \dot{a}^A_b} = -e^b_A . \tag{5.23}$$

The canonical Hamiltonian is then

$$
\begin{aligned}
H_C &= \int\limits_{\Sigma} dV_x \,\left(\pi^b_A \dot{a}^A_b\right) - L \\
&= H_0 + \int\limits_{\Sigma} dV_x \,\left(\pi^b_A \mathcal{D}_b \phi^A + g C^A_{BC} \pi^b_A a^C_b \phi^B\right) \,, \qquad (5.24)
\end{aligned}
$$

where we have defined H_0 as

$$
H_0 = \int\limits_{\Sigma} dV_x \,\left(\lambda\left(\frac{1}{2}\pi^b_A \pi^A_b + \frac{1}{4} f^A_{ab} f^{ab}_A\right) + \pi^b_A \mathcal{D}_b \phi^A + g C^A_{BC} \pi^b_A a^C_b \phi^B\right) \,. \qquad (5.25)
$$

This definition of H_0 will be relevant in the Hamiltonian BRST treatment. Since $\dot{\phi}^A$ is absent in the Lagrangian, $\pi^\phi_A \approx 0$ is the only primary constraint. Following the Dirac-Bergmann formalism, this constraint is added to the canonical Hamiltonian with the aid of a Lagrange multiplier v^A_ϕ to define a new Hamiltonian

$$
\tilde{H} = \int\limits_{\Sigma} dV_x \,\left(\lambda\left(\frac{1}{2}\pi^b_A \pi^A_b + \frac{1}{4} f^A_{ab} f^{ab}_A\right) + \pi^b_A \mathcal{D}_b \phi^A + g C^A_{BC} \pi^b_A a^C_b \phi^B + v^A_\phi \pi^\phi_A\right) \,, \quad (5.26)
$$

The canonical Poisson brackets of the theory are

$$
\begin{aligned}
\left[\phi^A(x), \pi^\phi_B(y)\right]_P &= \delta^A_B \delta(x,y) \\
\left[a^A_b(x), \pi^a_B(y)\right]_P &= \delta^A_B \delta^a_b \delta(x,y) \,. \qquad (5.27)
\end{aligned}
$$

The Dirac-Bergmann formalism now requires us to check $\dot{\pi}^\phi_A \approx 0\,.$ The time evolution of π^ϕ_A is evaluated through its Poisson bracket with $\tilde{H}$ by using a set of smearing functions ϵ^A as follows

$$\begin{aligned}
\int_\Sigma dV_y \epsilon^A(y)\dot{\pi}^\phi_A(y) &= \int_\Sigma dV_y \epsilon^A(y)\left[\pi^\phi_A(y), \tilde{H}\right]_P \\
&= \int_\Sigma dV_y \epsilon^A(y)\left[\pi^\phi_A(y), \int_\Sigma dV_x \pi^b_B(x)\mathcal{D}^x_b\phi^B(x) + gC^D_{BC}\pi^b_D(x)a^C_b(x)\phi^B(x)\right]_P \\
&= -\oint_{\partial\Sigma} da_y\, \epsilon^A(y)n^y_b\pi^b_A(y) + \int_\Sigma dV_y\, \epsilon^A(y)\left(\mathcal{D}^y_b\pi^b_A(y) - gC^D_{AC}\pi^b_D(y)a^C_b(y)\right) .
\end{aligned} \tag{5.28}$$

When the smearing functions ϵ^A are regular on the horizon, as we have assumed throughout this thesis, the surface integral is finite. Thus Eq. (5.28) provides

$$\int_\Sigma dV_x \epsilon^A(x)\Omega_{2A}(x) = \oint_{\partial\Sigma} da_x n^x_b \epsilon^A(x)\pi^b_A(x) - \int_\Sigma dV_x\, \epsilon^A(x)\left(\mathcal{D}^x_b\pi^b_A(x) + gC^D_{AC}\pi^b_D(x)a^C_b(x)\right) \approx 0 . \tag{5.29}$$

We can therefore express Ω_{2A} as

$$\Omega_{2A} = n_b\pi^b_A\Big|_{\mathcal{H}} - \mathcal{D}_b\pi^b_A + gC^D_{AC}\pi^b_D a^C_b \approx 0 . \tag{5.30}$$

The notation $\Big|_{\mathcal{H}}$ indicates that the corresponding term is the surface contribution to the constraint. Eq. (5.30) is a distribution which requires that we smear and integrate it over a given volume of the hypersurface Σ. Thus smearing Eq. (5.30) with functions ϵ^A which are regular at the horizons and integrating over Σ gives Eq. (5.29). Including Eq. (5.30) in the Hamiltonian with its own Lagrange multiplier, we can write the total Hamiltonian as

$$\begin{aligned}
H_T &= H_0 + \oint_{\partial\Sigma} da_x\, n_b(\phi^A + v^A_2)\pi^b_A + \int_\Sigma dV_x \left(v^A_\phi\pi^\phi_A - (\phi^A + v^A_2)\left(\mathcal{D}_b\pi^b_A - gC^D_{AC}\pi^b_D a^C_b\right)\right) \\
&= H_0 + \int_\Sigma dV_x \left((\phi^A + v^A_2)\Omega_{2A} + v^A_\phi\pi^\phi_A\right)
\end{aligned} \tag{5.31}$$

Using smearing functions which are regular on the horizon, it is straightforward to demonstrate that Ω_{2A} satisfies the following relation

$$[\Omega_{2A}(x), \Omega_{2B}(y)]_P = gC^D_{AB}\Omega_{2D}(x)\delta(x,y)\,, \tag{5.32}$$

from which it follows that

$$\int\limits_{\Sigma} dV_y\,[\epsilon(y)\Omega_{2A}(y), H_T]_P = \int\limits_{\Sigma} dV_y\epsilon(y)\left((\phi^A(y)+v_2^B(y))gC^D_{AB}\Omega_{2D}(y)\right) \approx 0\,. \tag{5.33}$$

Hence there are no further constraints. The Poisson bracket $\left[\pi^\phi_A\,, \Omega_{2B}\right]_P \approx 0$, confirms that all the constraints are first class.

From a general linear combination of the first class constraints $\epsilon_1^A\pi^\phi_A + \epsilon_2^A\Omega_{2A}$, we can find the following non-vanishing gauge transformations

$$\begin{aligned}\delta\phi^A &= \epsilon_1^A\\ \delta a_b^A &= \mathcal{D}_b\epsilon_2^A + gC^A_{BC}a_b^C\epsilon_2^B\\ \delta\pi^b_A &= gC^D_{AB}\pi^b_D\epsilon_2^B\end{aligned} \tag{5.34}$$

We can also determine the Lagrange multipliers. From the Poisson bracket $\left[\phi^A, H_T\right]_P$, we can identify $v_\phi^A = \dot{\phi}^A$. Likewise, the Poisson bracket $\left[a_b^A\,, H_T\right]_P$ agrees with Eq. (5.22) provided $\mathcal{D}_b v_2^A = 0$. We can therefore set $v_2^A = 0$ and $v_\phi^A = \dot{\phi}^A$ in H_T.

5.2.2 Hamiltonian BRST derivation of the Path Integral

As mentioned in Sec. [5.1.2], the Hamiltonian BRST treatment requires us to extend the phase space of the previous subsection to include the ghosts $\mathcal{C}^A$, the antighosts $\bar{\mathcal{C}}_A$ and their conjugate momenta $\left(\mathcal{P}_A\,, \bar{\mathcal{P}}^A\right)$. The ghost numbers of these fields and the canonical relations they satisfy are

$$\begin{aligned} \left[\bar{\mathcal{P}}^B(x), \bar{\mathcal{C}}_A(y)\right] = -\delta^B_A \delta(x,y) = \left[\mathcal{P}_A(x), \mathcal{C}^B(y)\right] \,, \\ \mathrm{gh}\left(\mathcal{C}^A\right) = 1 = \mathrm{gh}\left(\bar{\mathcal{P}}^A\right) \,, \\ \mathrm{gh}\left(\mathcal{P}_A\right) = -1 = \mathrm{gh}\left(\bar{\mathcal{C}}_A\right) \,, \end{aligned} \tag{5.35}$$

where all other brackets involving the ghosts and their momenta vanish. Following Eq. (5.13), we can construct the following BRST charge for the Yang-Mills field

$$Q_{\mathrm{BRST}} = \int\limits_{\Sigma} dV_x \left(\Omega_{2A}(x)\mathcal{C}^A(x) + \frac{1}{2} g \mathcal{P}_A(x) C^A_{BC} \mathcal{C}^B(x) \mathcal{C}^C(x) - i\bar{\mathcal{P}}^A(x) \pi^\phi_A(x) \right) . \tag{5.36}$$

Let us now denote the set of all the fields in the extended phase space by $\mu^\alpha \equiv \left(a^A_a, \pi^a_A, \mathcal{C}^A, \mathcal{P}_A, \bar{\mathcal{C}}_A, \bar{\mathcal{P}}^A, \phi^A, \pi^\phi_A\right)$, where $\alpha\,,\beta\,\cdots$ runs over all the degrees of freedom (spacetime and internal indices) of the fields. We will denote the BRST transformed fields by $s\mu^\alpha$, which can be derived from the Poisson bracket $[\mu^\alpha\,, Q_{\mathrm{BRST}}]_P = s\mu^\alpha$. Evaluating the Poisson brackets using a set of smearing functions which are regular at the horizons, we find the following expressions for $s\mu^\alpha$

$$\begin{aligned} s a^A_a = \mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a \,, \qquad & s\phi^A = -i\bar{\mathcal{P}}^A \,, \\ s\pi^a_A = -g C^D_{BA} \mathcal{C}^B \pi^a_D \,, \qquad & s\bar{\mathcal{C}}_A = i\pi^\phi_A \,, \\ s\mathcal{P}_A = -\Omega_{2A} - g\mathcal{P}_D C^D_{BA} \mathcal{C}^B \,, \qquad & s\mathcal{C}^A = -\frac{1}{2} g C^A_{BC} \mathcal{C}^B \mathcal{C}^C \,, \\ s\pi^\phi_A = 0 = s\bar{\mathcal{P}}^A \,. & \end{aligned} \tag{5.37}$$

These transformations have the same form as the BRST transformations of the Yang-Mills field on backgrounds without a boundary. We will now proceed to consider the gauge fixing function for the Yang-Mills field following Eq. (5.19)

$$\Psi = \int\limits_{\Sigma} dV_x \left(i\bar{\mathcal{C}}_A(x)\chi^A(x) + \mathcal{P}_A(x)\phi^A(x) \right) . \tag{5.38}$$

In flat spacetime, the gauge fixing term $\chi^A = \partial^a a_a^A + \pi_\phi^A$ is often used to recover the covariant action of the theory. On the spherically symmetric backgrounds we are considering, it will be useful to let χ^A involve the term $\mathcal{D}^a(\lambda a_a^A) + \frac{1}{2}\lambda\pi_\phi^A$. Since we are interested in the effect of horizons, we will also include surface terms at the horizons $n^a a_a^A\Big|_{\mathcal{H}}$ in the definition of χ^A, analogous to the horizon contributions present in Ω_{2A}. We thus choose

$$\chi^A = \mathcal{D}^a(\lambda a_a^A) - n^a a_a^A\Big|_{\mathcal{H}} + \frac{1}{2}\lambda\pi_\phi^A\,, \tag{5.39}$$

The BRST invariant Hamiltonian now follows from Eq. (5.16), which for the Yang-Mills field is given by

$$H_{eff} = H_0 - [\Psi, Q_{\mathrm{BRST}}]_P = \lambda\left(\frac{1}{2}\pi_A^a\pi_a^A + \frac{1}{4}f_{ab}^A f_A^{ab}\right) - [\Psi, Q_{\mathrm{BRST}}]_P\,. \tag{5.40}$$

As noted in Eq. (5.17), this allows us to further define the BRST invariant effective action by using the Legendre transform. We will now discuss a brief point on the Legendre transform which will be relevant in what follows. While the spacetime covariant measure is $\lambda dt dV_x$, functional derivatives in this thesis were defined with respect to the covariant measure of the spatial hypersurfaces dV_x. This leads to all non-vanishing momenta resulting from the Lagrangian of a given theory to involve a factor of λ. Thus no factors of λ appear in the definition of the canonical Hamiltonian of a theory resulting from the Legendre transform, as in Eq. (5.24) for the Yang-Mills field for example. However, in considering an effective action resulting from a BRST invariant Hamiltonian, we are completely free to rescale that part of the Legendre transform which involve the ghosts and Lagrange multipliers, and their conjugate momenta. This is due to the fact that these fields have no definition or normalization which follows from the Lagrangian of a given theory. We accordingly use this freedom while considering Eq. (5.17) and define

$$S_{eff} = \int dt \int_{\Sigma} dV_x \left(\dot{a}^A_a \pi^a_A + \lambda^{-1} \dot{\phi}^A \pi^\phi_A + \dot{\mathcal{C}}^A \mathcal{P}_A + \lambda^{-1} \dot{\bar{\mathcal{C}}}_A \bar{\mathcal{P}}^A - H_{eff} \right) , \tag{5.41}$$

We will justify these factors of λ by demonstrating that they provide the covariant action of the gauge fixed Yang-Mills field in this section.

The Poisson bracket $[\Psi, Q_{\text{BRST}}]_P$ can be evaluated from Eq. (5.38) and Eq. (5.36) and has the following result

$$\begin{aligned} [\Psi, Q_{\text{BRST}}]_P = - \Big(& \pi^\phi_A \chi^A + i \mathcal{P}^A \bar{\mathcal{P}}_A + \phi^A \left(\Omega_{2A} + g \mathcal{P}_D C^D_{BA} \mathcal{C}^B \right) \\ & + i \left(\lambda \mathcal{D}^a \bar{\mathcal{C}}_A + n_a \bar{\mathcal{C}}_A |_{\mathcal{H}} \right) \left(\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a \right) \Big) . \end{aligned} \tag{5.42}$$

Using Eq. (5.42), we define the path integral as in Eq. (5.18) which spans over all field variables μ^α

$$\begin{aligned} Z &= \int [D\mu] \, e^{iS_{eff}} \\ &= \int [D\mu] \exp \Bigg(i \int dt \int_{\Sigma} dV_x \Big[\dot{a}^A_a \pi^a_A + \lambda^{-1} \dot{\phi}^A \pi^\phi_A + \dot{\mathcal{C}}^A \mathcal{P}_A + \lambda^{-1} \dot{\bar{\mathcal{C}}}_A \bar{\mathcal{P}}^A - \pi^\phi_A \chi^A - i \mathcal{P}^A \bar{\mathcal{P}}_A \\ &\qquad - \lambda \left(\frac{1}{2} \pi^a_A \pi^A_a + \frac{1}{4} f^A_{ab} f^{ab}_A + i \mathcal{D}^a \bar{\mathcal{C}}_A \left(\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a \right) \right) \\ &\qquad - \phi^A \left(\Omega_{2A} + g \mathcal{P}_D C^D_{BA} \mathcal{C}^B \right) \Big] + \int dt \oint_{\partial \Sigma} da_x n_a \bar{\mathcal{C}}_A \left(\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a \right) \Bigg) , \end{aligned} \tag{5.43}$$

We can now integrate out $\bar{\mathcal{P}}^A$, $\mathcal{P}_A$ and π^a_A to find

$$
\begin{aligned}
Z = \int [D\tilde{\mu}] \exp \Bigg(i \int dt \int_\Sigma dV_x \, \lambda \left(\frac{1}{2} e^a_A e^A_a - \frac{1}{4} f^A_{ab} f^{ab}_A + \lambda \pi^\phi_A \left(\lambda^{-2} \dot{\phi}^A - \lambda^{-1} \mathcal{D}^a (\lambda a^A_a) - \frac{1}{2} \pi^A_\phi \right) \right) \\
+ \int dt \int_\Sigma dV_x \, \lambda \left(-\lambda^{-2} \dot{\bar{\mathcal{C}}}_A \left(\dot{\mathcal{C}}^A + g C^A_{BC} \mathcal{C}^B \phi^C \right) + \mathcal{D}^a \bar{\mathcal{C}}_A (\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a) \right) \\
+ i \int dt \oint_{\partial\Sigma} da_x \, n^a a^A_a \pi^\phi_A + \int dt \oint_{\partial\Sigma} da_x n_a \bar{\mathcal{C}}_A \left(\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a \right) \Bigg) \, ,
\end{aligned}
\tag{5.44}
$$

where only the fields $\left(a^A_a, \phi^A, \pi^\phi_A, \bar{\mathcal{C}}_A, \mathcal{C}^A \right)$ are contained in the measure $[D\tilde{\mu}]$. In Eq. (5.44), the term $\frac{1}{2} e^a_A e^A_a$ follows from Eq. (5.22) after integrating out π^a_A. We can further re-express Eq. (5.44) in terms of the effective projected Yang-Mills, gauge fixing and ghost actions in the following way

$$
Z = \int [D\tilde{\mu}] \, e^{i S_{eff}} = \int [D\tilde{\mu}] \, e^{i \left(S_{YM} + S_{gh} + S_{gf} \right)} \, , \tag{5.45}
$$

where

$$
S_{YM} = \int dt \int_\Sigma dV_x \, \lambda \left(\frac{1}{2} e^a_A e^A_a - \frac{1}{4} f^A_{ab} f^{ab}_A \right) \, , \tag{5.46}
$$

$$
S_{gf} = \int dt \int_\Sigma dV_x \, \lambda \pi^\phi_A \left(\lambda^{-2} \dot{\phi}^A - \lambda^{-1} \mathcal{D}^a (\lambda a^A_a) - \frac{1}{2} \pi^A_\phi \right) + \int dt \oint_{\partial\Sigma} da_x \, n^a a^A_a \pi^\phi_A \, , \tag{5.47}
$$

$$
\begin{aligned}
S_{gh} = i \int dt \int_\Sigma dV_x \, \lambda \left(\lambda^{-2} \dot{\bar{\mathcal{C}}}_A \left(\dot{\mathcal{C}}^A + g C^A_{BC} \mathcal{C}^B \phi^C \right) - \mathcal{D}^a \bar{\mathcal{C}}_A (\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a) \right) \\
- i \int dt \oint_{\partial\Sigma} da_x \, n^a \bar{\mathcal{C}}_A \left(\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a \right) \, .
\end{aligned}
\tag{5.48}
$$

The Schwarz inequality tells us nothing about the surface integrands in Eq. (5.47) and Eq. (5.48), since they are separately not gauge invariant and can have arbitrary norms. We can only state that Eq. (5.46), Eq. (5.47) and Eq. (5.48) are collectively BRST invariant. From Eq. (5.37), the BRST transformations are now

$$\begin{aligned} s a_a^A = \mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a_a^C \,, \qquad & s\phi^A = \dot{\mathcal{C}}^A + g C^A_{BC} \mathcal{C}^B \phi^C \,, \\ s\bar{\mathcal{C}}_A = i\pi^\phi_A \,, \qquad & s\mathcal{C}^A = -\frac{1}{2} g C^A_{BC} \mathcal{C}^B \mathcal{C}^C \,, \\ & s\pi^\phi_A = 0 \,. \end{aligned} \tag{5.49}$$

If the surface integral in Eq. (5.47) were absent, we could have performed the integration over π^ϕ_A in Eq. (5.45) to derive the following effective ghost action

$$S'_{gf} = \int dt \int_\Sigma dV_x \, \lambda \frac{1}{2} \left(\lambda^{-1} \mathcal{D}^a (\lambda a_a^A) - \lambda^{-2} \dot{\phi}^A \right)^2 \,. \tag{5.50}$$

The definitions $h_a^b A_b^A = a_a^A$ and $\xi^a A_a^A = \phi^A$, along with the projection operator in Eq. (5.1) allow us to further express Eq. (5.50) as

$$S'_{gf} = \int dV_4 \frac{1}{2} \left(\nabla_a A^a_A \right)^2 \,. \tag{5.51}$$

Thus Eq. (5.47) can be considered as the modification of the Lorenz gauge in the presence of horizons.

5.2.3 Renormalizability of the Yang-Mills field

The Zinn-Justin equation is a condition satisfied by the quantum effective action of a BRST invariant theory, which has been used to prove the perturbative renormalizability of nonabelian gauge theories, for example the Yang-Mills field [105, 106] and the non-Abelian two form [107], in flat spacetime. The purpose of this subsection is to extend the analysis of [106] to the Yang-Mills field on curved backgrounds with horizons. We will specifically be interested in the effect of surface integrals at the horizons on the renormalizability of the theory.

Let S describe a BRST invariant action involving the fields μ^α, whose transformations we denote by $s\mu^\alpha = F^\alpha$. As before, $\alpha, \beta, \cdots$ run over both spacetime and internal indices. For each of the fields μ^α we can introduce a set of anti-fields K_α, which have the same Grassmann parity as μ^α but the opposite ghost number of F^α. Thus if μ^α has parity $\epsilon(\mu^\alpha)$ and ghost number $\mathrm{gh}(\mu^\alpha) = \gamma^\alpha$, then $\mathrm{gh}(K^\alpha) = -\gamma^\alpha - 1$ and $\epsilon(K_\alpha) = \epsilon(\mu^\alpha)$. We can now consider the path integral

$$Z[J,K] = \int \left[\mathcal{D}\mu^B\right] \exp\left(iS + i\int dt \int\limits_\Sigma dV_x \mu^\alpha J_\alpha + i\int dt \int\limits_\Sigma dV_x F^\alpha K_\alpha \right) , \quad (5.52)$$

The terms contained in the parenthesis of Eq. (5.52) represent the action in the presence of sources, which is invariant under BRST transformations and the interchange of the fields and anti-fields.

In this subsection, we will consider variations over the spacetime which preserve the foliation of the background. We denote these functional variations with respect to the fields by $\bar{\delta}$, which satisfies

$$\frac{\bar{\delta}\mu^\alpha(x)}{\bar{\delta}\mu^\beta(x')} = \delta^\alpha_\beta \delta\left(\vec{x}, \vec{x}'\right)\delta(t,t') \,. \quad (5.53)$$

While $\delta\left(\vec{x}, \vec{x}'\right)$ is the delta function defined on Σ, $\delta\left(\vec{x}, \vec{x}'\right)\delta(t,t')$ is the delta function defined on $\mathcal{M} = \Sigma \times \mathbb{R}$. Thus for any well defined function $f(\vec{x}, t)$ of the spacetime, we have

$$\int dt \int\limits_\Sigma dV_x \delta\left(\vec{x}, \vec{x}'\right)\delta(t,t') f(\vec{x},t) = f(\vec{x}', t') \,. \quad (5.54)$$

In the case of Grassmannian fields, we will need to consider 'left' and 'right' variations as defined in Eq. (2.7), only now with δ replaced with $\bar{\delta}$.

Given the path integral in the presence of sources, we can define the 'connected vacuum persistence amplitude' $W[J,K] = -i\ln Z[J,K]$. The expectation value of the fields in the presence of sources $\langle\mu_\alpha\rangle_{J,K}$ can be determined through the following functional variation

$$\langle\mu^\alpha\rangle_{J,K} = i\frac{1}{Z[J,K]}\frac{\bar{\delta}_R Z[J,K]}{\bar{\delta}J_\alpha} = \frac{\bar{\delta}_R W[J,K]}{\bar{\delta}J_\alpha}\,. \tag{5.55}$$

Let us now also define $J_{\alpha\,;\mu,K}$ as the value of the current for which $\langle\mu^\alpha\rangle_{J,K} = \mu^\alpha_{J,K}$ is the expectation value resulting from Eq. (5.55). The quantum effective action is then defined by

$$\Gamma[\mu,K] = W[J_{\mu,K},K] - \int dt \int\limits_\Sigma dV_x \mu^\alpha_{J,K} J_{\alpha\,;\mu,K}\,, \tag{5.56}$$

If the action S and the path integral measure appearing in Eq. (5.52) are both invariant under the BRST transformations $s\mu^\alpha = F^\alpha$, then the path integral in Eq. (5.52) is invariant under the BRST transformations provided

$$\int dt \int\limits_\Sigma dV_x \langle F^\alpha\rangle_{J,K}\frac{\bar{\delta}_L\Gamma[\mu,K]}{\bar{\delta}\mu^\alpha} = 0\,. \tag{5.57}$$

These are the Slavnov-Taylor identities. For the Maxwell field and other gauge theories invariant under linear transformations, Eq. (5.57) is equivalent to the statement that $\Gamma[\mu,K]$ is invariant under $\mu^\alpha \to \mu^\alpha + \epsilon\langle F^\alpha\rangle_{J,K}$ where ϵ is an infinitessimal parameter. However, the BRST transformations for the fields a^A_a, ϕ^a and $\mathcal{C}^A$ are not linear in the fields. In the case of the Yang-Mills field, we can instead use the Zinn-Justin equation

$$(\Gamma,\Gamma) = 0\,, \tag{5.58}$$

where the antibracket (F,G) has the following definition for any two functionals F and G

$$(F,G) = \int dt \int\limits_\Sigma dV_x \left(\frac{\bar{\delta}_R F[\mu,K]}{\bar{\delta}\mu^\alpha(x)}\frac{\bar{\delta}_L G[\mu,K]}{\bar{\delta}K_\alpha(x)} - \frac{\bar{\delta}_R F[\mu,K]}{\bar{\delta}K_\alpha(x)}\frac{\bar{\delta}_L G[\mu,K]}{\bar{\delta}\mu^\alpha(x)}\right)\,. \tag{5.59}$$

The derivation of Eq. (5.58) follows from Eq. (5.56) and Eq. (5.57). The variation of Eq. (5.56) with respect to K^α gives

$$\begin{aligned}\frac{\bar{\delta}_R \Gamma[\mu, K]}{\bar{\delta} K_\alpha} &= \left.\frac{\bar{\delta}_R W[J, K]}{\bar{\delta} K_\alpha}\right|_{J=J_{\mu,K}} + \int dt \int_\Sigma dV_x \left.\frac{\bar{\delta}_R W[J, K]}{\bar{\delta} J_\beta}\right|_{J=J_{\mu,K}} \frac{\bar{\delta}_R J_{\beta\,;\mu,K}}{\bar{\delta} K_\alpha} \\ &\quad - \int dt \int_\Sigma dV_x \mu^\beta_{J,K} \frac{\bar{\delta}_R J_{\beta\,;\mu,K}}{\bar{\delta} K_\alpha} \\ &= \left.\frac{\bar{\delta}_R W[J, K]}{\bar{\delta} K_\alpha}\right|_{J=J_{\mu,K}} . \end{aligned} \tag{5.60}$$

We made use of Eq. (5.55) in deriving the result of Eq. (5.60). It now follows that

$$\left.\frac{\bar{\delta}_R W[J, K]}{\bar{\delta} K_\alpha}\right|_{J=J_{\mu,K}} = \langle F^\alpha \rangle_{J,K} = \frac{\bar{\delta}_R \Gamma[\mu, K]}{\bar{\delta} K_\alpha} . \tag{5.61}$$

Eq. (5.57) and Eq. (5.61) together imply

$$\int dt \int_\Sigma dV_x \frac{\bar{\delta}_R \Gamma[\mu, K]}{\bar{\delta} K_\alpha} \frac{\bar{\delta}_L \Gamma[\mu, K]}{\bar{\delta} \mu^\alpha} = 0 . \tag{5.62}$$

Since the interchange of fields and anti-fields is a symmetry of the effective action, this leads to the Zinn-Justin equation given in Eq. (5.58).

We will now use the Zinn-Justin equation to demonstrate that the quantum effective action is invariant under renormalized BRST transformations which have the same form as the classical Yang-Mills action. This will further allow us to determine the general form of the renormalizable quantum effective action.

Let us define the following actions on the basis of their sources

$$S[\mu] = S + \int dt \int dV \mu^\alpha J_\alpha , \tag{5.63}$$

$$S[\mu, K] = S[\mu] + \int dt \int dV_x F^\alpha K_\alpha . \tag{5.64}$$

We assume that $S[\mu, K]$ can be written as the sum of a renormalized action

$$S_R[\mu, K] = S_R[\mu, 0] + \int dt \int dV_x F^\alpha(\mu, x) K_\alpha \tag{5.65}$$

and a term $S_\infty[\mu, K]$ containing counterterms intended to cancel loop infinities. $S_R[\mu, K]$ and $S_\infty[\mu, K]$ must have the same symmetries as $S[\mu, K]$ to ensure that infinite contributions to $\Gamma[\mu, K]$ can be cancelled by counterterms in $S_\infty[\mu, K]$. We can now expand $\Gamma[\mu, K]$ in terms of the loop expansion parameter $\hbar$,

$$\Gamma[\mu, K] = \sum_{N=0}^{\infty} \hbar^{N-1} \Gamma_N[\mu, K] \,, \tag{5.66}$$

where $\Gamma_0[\mu, K] = S_R[\mu, K]$. The Zinn-Justin equation can thus be written order-by order for each N as

$$\sum_{N'=0}^{N} (\Gamma_{N'}, \Gamma_{N-N'}) = 0 \,. \tag{5.67}$$

This expansion includes the counterterms contained in $S_\infty[\mu, K]$ needed to cancel the sub-divergences at any given loop order N. If at some given N all infinities appearing up to $N-1$ loop order have been cancelled by counterterms in S_∞, then the only remaining infinities in Eq. (5.67) come from Γ_N. In this case, the infinite part $\Gamma_{N,\infty}$ satisfies

$$(S_R, \Gamma_{N,\infty}) = 0 \,. \tag{5.68}$$

For the Yang-Mills field, it can be shown that $\Gamma_{N,\infty}$ has a linear dependence on the anti-fields K_α and therefore can be written as

$$\Gamma_{N,\infty}[\mu, K] = \Gamma_{N,\infty}[\mu, 0] + \int dt \int_\Sigma dV_x \mathscr{F}^\alpha_N[\mu, x] K_\alpha(x) \,, \tag{5.69}$$

where $\mathscr{F}^\alpha_N[\mu, x]$ can have an arbitrary dependence on the fields μ^α and the coordinates. To see this, let us first consider the mass dimension and ghost numbers of the anti-fields. Denoting the mass dimension and ghost number of all the fields μ^α

by d^α and γ^α respectively, the F^α are fields of mass dimension $d^\alpha + 1$ and ghost number $\gamma^\alpha + 1$. Thus the anti-fields K^α are of mass dimension $3 - d^\alpha$ and ghost number $-\gamma^\alpha - 1$. The effective action must have zero mass dimension and ghost number. From the consideration of mass dimension alone, the Lagrangian of the quantum effective action can at most be quadratic in K_α. We can also note that apart from the anti-field corresponding to the field $\bar{\mathcal{C}}_A$, which we denote here by $K^{\bar{\mathcal{C}}}_A$, all other anti-fields have a non-vanishing ghost number. Thus from power-counting arguments, we see that the quantum effective action could only be quadratic in $K^{\bar{\mathcal{C}}}_A$. However, from the BRST transformation of $\bar{\mathcal{C}}_A$ and its relation to the symmetries of the quantum effective action, i.e.

$$\frac{\delta_L \Gamma_{N,\infty}}{\delta K^{\bar{\mathcal{C}}}_A} = \langle F^A_{\bar{\mathcal{C}}} \rangle_{J,K} = s\bar{\mathcal{C}}_A = i\lambda \pi^\phi_A \,, \tag{5.70}$$

it follows that $\Gamma_{N,\infty}$ can only depend linearly on $K^{\bar{\mathcal{C}}}_A$. Thus $\Gamma_{N,\infty}$ has a linear dependence on all the anti-fields as indicated in Eq (5.69).

Using Eq. (5.65) and Eq. (5.69) in Eq. (5.68), we can find the following equations at zeroth-order and first-order in K^α

$$\int dt \int\limits_{\Sigma} dV_x \left[F^\alpha \frac{\bar{\delta}_L \Gamma_{N,\infty}[\mu, 0]}{\bar{\delta}\mu^\alpha} - \frac{\bar{\delta}_R S_R[\mu, 0]}{\bar{\delta}\mu^\alpha} \mathscr{F}^\alpha_N \right] = 0 \,, \tag{5.71}$$

$$\int dt \int\limits_{\Sigma} dV_x \left[F^\alpha \frac{\bar{\delta}_L \mathscr{F}^\alpha_N}{\bar{\delta}\mu^\alpha} - \frac{\bar{\delta}_R F^\alpha}{\bar{\delta}\mu^\alpha} \mathscr{F}^\alpha_N \right] = 0 \,. \tag{5.72}$$

These equations suggest the following definition of $\Gamma^{(\epsilon)}_N[\mu]$

$$\Gamma^{(\epsilon)}_N[\mu] = S_R[\mu, 0] + \epsilon \Gamma_{N,\infty}[\mu, 0] \,, \tag{5.73}$$

where ϵ is an infinitesimal parameter. Eq. (5.71) now implies that $\Gamma^{(\epsilon)}_N[\mu]$ is invariant under the following transformation

$$s_R \mu^A(x) = F^{(\epsilon)A}_N(x) \,, \tag{5.74}$$

where

$$F_N^{(\epsilon)A}(x) = F^A(x) + \epsilon \mathscr{F}_N^A(x)\,. \tag{5.75}$$

Eq. (5.72) tells us that the renormalized transformations in Eq. (5.74) are nilpotent, i.e. $s_R^2 = 0$. These nilpotent transformations can be easily shown to be of the same form as the BRST transformations which leave the tree level action of the theory invariant. As the BRST transformations for the fields $\bar{\mathcal{C}}_A$ and π_A^ϕ are linear, they are not affected and we have

$$s_R \bar{\mathcal{C}}_A = i\pi_A^\phi \qquad s_R \pi_A^\phi = 0\,. \tag{5.76}$$

The renormalized transformations of the fields $a_A^A\,, \phi^A$ and $\mathcal{C}_A$ must have the same Lorentz transformation and ghost number properties of the original transformations $F^A = s\mu^A$. Hence the most general form of Eq. (5.74) can be written as

$$\begin{aligned}
s_R a_a^A &= G_B^A \mathcal{D}_a \mathcal{C}^B + g D_{BC}^A \mathcal{C}^B a_a^C\,,\\
s_R \phi^A &= H_B^A \dot{\mathcal{C}}^B + g B_{BC}^A \mathcal{C}^B \phi^C\,,\\
s_R \mathcal{C}^A &= -\frac{1}{2} g E_{BC}^A \mathcal{C}^B \mathcal{C}^C\,,
\end{aligned} \tag{5.77}$$

where G_B^A, H_B^A, D_{BC}^A, B_{BC}^A and E_{BC}^A are as yet undertermined constants. The requirement of nilpotence, $s_R^2 \mu^\alpha = 0$, leads to the following relation

$$E_{BC}^A E_{FG}^B + E_{BG}^A E_{CF}^B + E_{BF}^A E_{GC}^B = 0\,, \tag{5.78}$$

when applied to $\mathcal{C}^A$, and

$$\begin{aligned}
&g\left(-G_B^A E_{FG}^B + G_G^B D_{FB}^A\right)\mathcal{C}^F \mathcal{D}_a \mathcal{C}^G - \frac{g^2}{2}\left(D_{BC}^A E_{FG}^B + D_{BG}^A D_{FC}^B - D_{BF}^A D_{GC}^B\right) a_a^C \mathcal{C}^F \mathcal{C}^G\,,\\
&g\left(-H_B^A E_{FG}^B + H_G^B B_{FB}^A\right)\mathcal{C}^F \dot{\mathcal{C}}^G - \frac{g^2}{2}\left(B_{BC}^A E_{FG}^B + B_{BG}^A B_{FC}^B - B_{BF}^A B_{GC}^B\right) \phi^C \mathcal{C}^F \mathcal{C}^G\,,
\end{aligned} \tag{5.79}$$

when applied to a_a^A and ϕ^A respectively. We see that Eq. (5.78) is satisfied when E^A_{BC} is proportional to C^A_{BC}. Likewise, Eq (5.79) is satisfied when D^A_{BC} and B^A_{BC} are proportional to C^A_{BC}, and when G^A_B and H^A_B are proportional to δ^A_B. This allows us to write the renormalized transformations in Eq. (5.77) and Eq. (5.76) using an arbitrary multiplicative constant $\mathscr{S}$ as

$$\begin{aligned} s_R a_a^A &= \mathscr{S}\left(\mathcal{D}_a\mathcal{C}^A + gC^A_{BC}\mathcal{C}^B a_a^C\right)\,, \qquad s_R\phi^A = \mathscr{S}\left(\dot{\mathcal{C}}^A + gC^A_{BC}\mathcal{C}^B\phi^C\right)\,, \\ s_R\mathcal{C}^A &= -\mathscr{S}\left(\frac{1}{2}gC^A_{BC}\mathcal{C}^B\mathcal{C}^C\right)\,, \qquad s_R\bar{\mathcal{C}}_A = i\pi^\phi_A\,, \\ & s_R\pi^\phi_A = 0\,. \end{aligned} \tag{5.80}$$

Thus by allowing for a general set of transformations of the Yang-Mills and ghost fields which increase the ghost number and mass dimension by 1, the Zinn-Justin equation tells us that the renormalized transformations have a similar form as the BRST transformations which leave the tree level action invariant.

We will now consider a quantum effective action comprising of both volume and surface integrals, which we denote by

$$\Gamma^{(\epsilon)}_N[\mu] = \int dt \int_\Sigma dV_x \mathcal{L}^\epsilon + \int dt \oint_{\partial\Sigma} da_x n^a \mathscr{L}^\epsilon_a\,, \tag{5.81}$$

where $\mathcal{L}^\epsilon$ and $n^a\mathscr{L}^\epsilon_a$ correspond to the bulk and horizon Lagrangian densities, respectively.

We will assume $\mathcal{L}^\epsilon$ involves the free Yang-Mills Lagrangian given in Eq. (5.46) with renormalized coefficients. In particular, we do not assume any contribution to $n^a\mathscr{L}^\epsilon_a$ from the free Yang-Mills part of the quantum effective action. Let us now consider terms in the quantum effective action which involve π^ϕ_A and the ghosts, namely the gauge fixing and ghost contributions to $\Gamma^{(\epsilon)}_N[\mu]$. All terms in $\mathcal{L}^\epsilon$ must have mass dimension 4, while all terms in $n^a\mathscr{L}^\epsilon_a$ must have mass dimension 3. Any term which involves ghosts must also involve an equal number of antighosts to ensure a zero ghost number contribution in the quantum effective action. It is easy to verify that the only possible contributions to $\Gamma_N^{(\epsilon)}[\mu]$ which involve the ghosts and satisfy these requirements are those already contained in Eq. (5.48), up to arbitrary constants. We thus assume that $\mathcal{L}^\epsilon$ and $n^a\mathscr{L}_a^\epsilon$ contain terms present in Eq. (5.48), with arbitrary constant coefficients whose relations are to be determined.

Finally for the gauge fixing contribution to $\Gamma_N^{(\epsilon)}[\mu]$, we need to consider all possible terms which include π_A^ϕ, which we recall has mass dimension 2. As in the case of the ghost contribution, we will assume that the terms contained in Eq. (5.47) also appear in $\Gamma_N^{(\epsilon)}[\mu]$, but now with arbitrary constant coefficients. There exists only one other term involving π_A^ϕ which is allowed on the basis of symmetry and power counting arguments, but which is not contained in Eq. (5.47). This is the term $\pi_A^\phi\left(a_a^B a^{Ca} + \phi^B\phi^C\right)$, which we will consider in $\mathcal{L}^\epsilon$ with its own constant coefficient.

Thus $\mathcal{L}^\epsilon$ and $n^a\mathscr{L}_a^\epsilon$ have the expressions

$$\begin{aligned}\mathcal{L}^\epsilon &= \mathcal{L}_{YM} + b_1\pi_\phi^A\pi_A^\phi + \lambda b_2\pi_\phi^A\left(\mathcal{D}_b\left(\lambda a_A^b\right) - \lambda^{-1}\dot{\phi}_A\right) + i\lambda\mathcal{Z}_\mathcal{C}\left(\mathcal{D}_a\bar{\mathcal{C}}_A\mathcal{D}^a\mathcal{C}^A - \lambda^{-2}\dot{\bar{\mathcal{C}}}_A\dot{\mathcal{C}}_A\right)\\ &\quad + i\lambda d_{BC}^A\left(\mathcal{D}_a\bar{\mathcal{C}}_A\mathcal{C}^B a^{Ca} - \lambda^{-2}\dot{\bar{\mathcal{C}}}_A\mathcal{C}^B\phi^C\right) + \frac{1}{2}e_{BC}^A\pi_A^\phi\left(a_a^B a^{Ca} + \phi^B\phi^C\right)\,,\\ \mathscr{L}_a^\epsilon &= i\bar{\mathcal{Z}}_\mathcal{C}\bar{\mathcal{C}}_A\mathcal{D}_a\mathcal{C}^A + il_{BC}^A\bar{\mathcal{C}}_A\mathcal{C}^B a_a^C + b_3\pi_A^\phi a_a^A\,,\end{aligned} \tag{5.82}$$

where b_1, b_2, b_3, $\bar{\mathcal{Z}}_\mathcal{C}$ and $\mathcal{Z}_\mathcal{C}$ are undetermined constants, d_{BC}^A and l_{BC}^A are totally antisymmetric constants, and e_{BC}^A is a constant which is symmetric in B and C. The variation of Eq. (5.81) under Eq. (5.80) is given by

$$
\begin{aligned}
s_R\Gamma_N^{(\epsilon)}[\mu] = \int dt \int_\Sigma dV_x \Big(& \mathscr{S} e^A_{BC}\pi^\phi_A \left(\left(\mathcal{D}_a\mathcal{C}^B + gC^B_{DE}\mathcal{C}^D a^E_a\right) a^{Ca} + (\dot{\mathcal{C}}^B + gC^B_{DE}\mathcal{C}^D\phi^E)\phi^C \right) \\
& + \frac{ig}{2}\mathscr{S}\left(\lambda^{-1}\dot{\bar{\mathcal{C}}}_A\mathcal{C}^D\mathcal{C}^E\phi^C - \lambda(\mathcal{D}^a\bar{\mathcal{C}}_A)\mathcal{C}^D\mathcal{C}^E a^C_a\right)\left(d^A_{BC}C^B_{DE} + d^A_{BD}C^B_{EC} + d^A_{BE}C^B_{CD}\right) \\
& + i\mathscr{S}\left(\lambda(\mathcal{D}_a\bar{\mathcal{C}}_A)(\mathcal{D}^a\mathcal{C}^B)\mathcal{C}^C - \lambda^{-1}\dot{\bar{\mathcal{C}}}_A\dot{\mathcal{C}}^B\mathcal{C}^C\right)\left(d^A_{BC} - g\mathcal{Z}_\mathcal{C}C^A_{BC}\right) \\
& + \pi^\phi_A\left(\lambda\mathcal{D}^a\left(\lambda\mathcal{C}^B a^C_a\right) - \pounds_\xi\left(\mathcal{C}^B\phi^C\right)\right)\left(\mathscr{S}gb_2C^A_{BC} + d^A_{BC}\right) \\
& + \pi^\phi_A\left(\lambda\mathcal{D}^a\lambda\mathcal{D}_a\mathcal{C}^A - \ddot{\mathcal{C}}^A\right)\left(\mathscr{S}b_2 + \mathcal{Z}_\mathcal{C}\right)\Big) \\
- \int dt \oint_{\partial\Sigma} da_x n^a \Big(& \frac{ig}{2}\mathscr{S}\bar{\mathcal{C}}_A\mathcal{C}^D\mathcal{C}^E a^C_a(l^A_{BC}C^B_{DE} + l^A_{BD}C^B_{EC} + l^A_{BE}C^B_{CD}) + \pi^\phi_A\mathcal{D}_a\mathcal{C}^A(\bar{\mathcal{Z}}_\mathcal{C} - \mathscr{S}b_3) \\
& + i\mathscr{S}\bar{\mathcal{C}}_A(\mathcal{D}_a\mathcal{C}^B)\mathcal{C}^C\left(g\bar{\mathcal{Z}}_\mathcal{C}C^A_{BC} - l^A_{BC}\right) + \lambda\pi^\phi_A\mathcal{C}^B a^C_a\left(l^A_{BC} - \mathscr{S}gb_3C^A_{BC}\right)\Big)
\end{aligned}
\tag{5.83}
$$

It can be observed that Eq. (5.83) vanishes provided

$$
\begin{aligned}
b_2 = -\frac{1}{\mathscr{S}}\mathcal{Z}_\mathcal{C}\,, \qquad & b_3 = \frac{1}{\mathscr{S}}\bar{\mathcal{Z}}_\mathcal{C}\,, \\
d^A_{BC} = g\mathcal{Z}_\mathcal{C}C^A_{BC}\,, \qquad & l^A_{BC} = g\bar{\mathcal{Z}}_\mathcal{C}C^A_{BC}\,, \\
e^A_{BC} = 0\,. &
\end{aligned}
\tag{5.84}
$$

Denoting the renormalized coefficient in the Yang-Mills action by $\mathcal{Z}_A$ and using Eq. (5.84), we can write Eq. (5.81) in the following way

$$
\begin{aligned}
\Gamma_N^{(\epsilon)}[\mu] = \int dt \int_\Sigma dV_x\lambda \Big(& \mathcal{Z}_A\left(\frac{1}{2}e^a_A e^A_a - \frac{1}{4}f^A_{ab}f^{ab}_A\right) + \pi^\phi_A\left(\frac{\mathcal{Z}_\mathcal{C}}{\mathscr{S}}\left(\lambda^{-2}\dot{\phi}^A - \lambda^{-1}\mathcal{D}^b\left(\lambda a^A_b\right)\right) + b_1\pi^A_\phi\right) \\
& + i\mathcal{Z}_\mathcal{C}\left(\mathcal{D}^a\bar{\mathcal{C}}_A\left(\mathcal{D}_a\mathcal{C}^A + igC^A_{BC}\mathcal{C}^B a^C_a\right) - \lambda^{-2}\dot{\bar{\mathcal{C}}}_A\left(\dot{\mathcal{C}}_A + igC^A_{BC}\mathcal{C}^B\phi^C\right)\right)\Big) \\
+ \int dt \oint_{\partial\Sigma} da_x n^a & \left(i\bar{\mathcal{Z}}_\mathcal{C}\bar{\mathcal{C}}_A\left(\mathcal{D}_a\mathcal{C}^A + igC^A_{BC}\mathcal{C}^B a^C_a\right) + \frac{1}{\mathscr{S}}\bar{\mathcal{Z}}_\mathcal{C}\pi^\phi_A a^A_a\right).
\end{aligned}
\tag{5.85}
$$

Eq. (5.85), apart from the presence of new constants, simply involve the effective actions given in Eq. (5.46), Eq. (5.47) and Eq. (5.48). We can freely choose the new constants such that $\Gamma_N^{(\epsilon)} = S_R$, in which case $\Gamma_{N,\infty} = 0$ and hence the theory is renormalizable. Thus surface integrals at the horizons, specifically those appearing in Eq. (5.47) and Eq. (5.48), do not affect the renormalizability of the Yang-Mills field on curved backgrounds.

5.3 Scalar Electrodynamics on spherically symmetric backgrounds

Having investigated the Yang-Mills field and its renormalizability on spherically symmetric backgrounds with horizons in the previous section, we now turn our attention to the effect of horizons on matter fields coupled to gauge theories. This will be considered through the example of scalar electrodynamics in the following.

5.3.1 Derivation of the Gauss law constraint

This subsection will follow the treatment provided in Chapter 3. The action for the complex scalar field minimally coupled to the electromagnetic field is given by

$$S_{SED} = -\int dV_4 \left(D_a\Phi(D_b\Phi)^* g^{ab} + m^2\Phi\Phi^* + \tfrac{1}{4}F_{ab}F_{cd}g^{ac}g^{bd} \right) \,, \tag{5.86}$$

where dV_4 is the four-dimensional volume form of the manifold with metric g_{ab}, Φ is a complex scalar field, $D_a = \partial_a + iqA_a$ is the gauge covariant derivative and $F_{ab} = \partial_a A_b - \partial_b A_a$ is the electromagnetic field strength tensor. We can now consider this action on $\Sigma \times \mathbb{R}$. Using the projection operator in Eq. (3.4) we have $dV_4 = \lambda dV_x$. We will also define the projected fields $a_a = h_a^b A_b$, $\phi = \xi^a A_a$, $e_d = -\lambda^{-1}\xi^c F_{cd}$,

$\bar{D}_a = \partial_a + iqa_a$, $D_0 = \pounds_\xi + iq\phi$ and $f_{ab} = h_a^c h_b^d F_{cd}$. Time derivatives are defined as the Lie derivative with respect to ξ^a, as defined in Eq. (3.9). In particular

$$\pounds_\xi a_b = \dot{a}_b = -\lambda e_b + \mathcal{D}_b \phi \,, \tag{5.87}$$

and likewise for the other fields. By projecting Eq. (5.86), we find

$$S_{SED} = \int dt \int_\Sigma dV_x \lambda \left(\lambda^{-2} D_0 \Phi (D_0 \Phi)^* - h^{ab} \bar{D}_a \Phi \left(\bar{D}_b \Phi^* \right) - m^2 \Phi \Phi^* - \frac{1}{4} f_{ab} f^{ab} + \frac{1}{2} e_a e^a \right) \,, \tag{5.88}$$

Denoting the momenta conjugate to $a_b\,, \phi\,, \Phi$ and Φ^* by $\pi^b, \pi^\phi\,, \Pi$ and Π^* respectively, we have

$$\begin{aligned} \pi^b &= \frac{\partial L_{SED}}{\partial \dot{a}_b} = -e^b \,, & \pi^\phi &= \frac{\partial L_{SED}}{\partial \dot{\phi}} = 0 \,, \\ \Pi &= \frac{\partial L_{SED}}{\partial \dot{\Phi}} = \lambda^{-1} (D_0 \Phi)^* \,, & \Pi^* &= \frac{\partial L_{SED}}{\partial \dot{\Phi}^*} = \lambda^{-1} D_0 \Phi \,. \end{aligned} \tag{5.89}$$

The canonical Hamiltonian can be constructed from the Legendre transform

$$\begin{aligned} H_C &= \int_\Sigma dV_x \left(\pi^b \dot{a}_b + \Pi \dot{\Phi} + \Pi^* \dot{\Phi}^* \right) - L \\ &= H_0 + \int_\Sigma dV_x \left(\pi^b \mathcal{D}_b \phi + iq\phi \left(\Phi^* \Pi^* - \Phi \Pi \right) \right) \end{aligned} \tag{5.90}$$

where H_0 in Eq. (5.90) is given by

$$H_0 = \int_\Sigma dV_x \, \lambda \left(\frac{1}{2} \pi^b \pi_b + \frac{1}{4} f_{ab} f^{ab} + \Pi \, \Pi^* + m^2 \Phi \Phi^* + \bar{D}_a \Phi \left(\bar{D}^a \Phi \right)^* \right) \tag{5.91}$$

The definition of H_0 given above will be relevant in considering the BRST treatment of this theory. Following the Dirac-Bergmann algorithm, we add the primary constraint to the canonical Hamiltonian to define a new Hamiltonian

$$\tilde{H} = H_C + \int_\Sigma dV_x \, v_\phi \pi^\phi \,, \tag{5.92}$$

where v_ϕ is an undetermined multiplier. From Eq. (2.14), we have the following canonical Poisson brackets of the theory

$$\begin{aligned} \left[\phi(x), \pi^\phi(y)\right]_P &= \delta(x,y)\,, \qquad & [a_b(x), \pi^a(y)]_P &= \delta^a_b \delta(x,y)\,, \\ [\Phi(x), \Pi(y)]_P &= \delta(x,y)\,, \qquad & [\Phi^*(x), \Pi^*(y)]_P &= \delta(x,y)\,. \end{aligned} \tag{5.93}$$

The consistency check of the primary constraint $\dot{\pi}^\phi \approx 0$, is evaluated through the Poisson bracket of π^ϕ and $\tilde{H}$ with the help of a smearing function ϵ as follows,

$$\begin{aligned} \int\limits_\Sigma dV_y \epsilon(y) \dot{\pi}^\phi(y) &= \int\limits_\Sigma dV_y \epsilon(y) \left[\pi^\phi(y), \tilde{H}\right]_P \\ &= \int\limits_\Sigma dV_y \epsilon(y) \left[\pi^\phi(y), \int\limits_\Sigma dV_x \left(\pi^b(x) \mathcal{D}^x_b \phi(x) + iq\phi(x) \left(\Phi^*(x)\Pi^*(x) - \Phi(x)\Pi(x)\right)\right)\right]_P \\ &= -\oint\limits_{\partial\Sigma} da_y\, \epsilon(y) n^y_b \pi^b(y) + \int\limits_\Sigma dV_y\, \epsilon(y) \left(\mathcal{D}^y_b \pi^b(y) - iq\left(\Phi^*(y)\Pi^*(y) - \Phi(y)\Pi(y)\right)\right)\,, \end{aligned} \tag{5.94}$$

where n_b is the unit spatial normal at the horizons which points in the direction of increasing time. Since the smearing function ϵ is regular at the horizons, we find the following modified Gauss law constraint

$$\int\limits_\Sigma dV_y\, \epsilon(y) \Omega_2(y) = \oint\limits_{\partial\Sigma} da_y\, n^y_b \epsilon(y) \pi^b(y) - \int\limits_\Sigma dV_y\, \epsilon(y) \left(\mathcal{D}^y_b \pi^b(y) - iq\left(\Phi^*(y)\Pi^*(y) - \Phi(y)\Pi(y)\right)\right)\,, \tag{5.95}$$

or equivalently

$$\Omega_2 = n_b \pi^b \Big|_{\mathcal{H}} - \mathcal{D}_b \pi^b + iq\left(\Phi^*\Pi^* - \Phi\Pi\right) \approx 0\,, \tag{5.96}$$

The vertical bar on the first term denotes that its contribution is restricted to the horizon(s) of the spacetime. Thus, by smearing Eq. (5.96) with a regular function ϵ and integrating over Σ, we find the expression given in Eq. (5.95). Including the constraint Eq. (5.96) in the Hamiltonian with its own Lagrange multiplier (v_2), we can write the total Hamiltonian as

$$H_T = H_0 + \int_\Sigma dV_x \left(v_\phi \pi^\phi - (v_2(x) + \phi(x))\mathcal{D}_b^x \pi^b(x) + iq\left(\Phi(x)^*\Pi(x)^* - \Phi(x)\Pi(x)\right)\right)$$
$$+ \oint_{\partial\Sigma} da_x n_b^x (v_2(x) + \phi(x))\pi^b(x)$$
$$H_T = H_0 + \int_\Sigma dV_x \left((v_2(x) + \phi(x))\Omega_2(x) + v_\phi(x)\pi^\phi(x)\right) . \tag{5.97}$$

It is straightforward to verify that $\dot{\Omega}_2 \approx 0$, which reveals that there are no further constraints of the theory. Since $\left[\pi^\phi, \Omega_2\right]_P = 0$, the constraints are first class and generate gauge transformations of the fields. These transformations follow from the Poisson brackets of the fields with the general linear combination of the first class constraints, $\epsilon_1\pi^\phi + \epsilon_2\Omega_2$

$$\begin{aligned} \delta\phi &= \epsilon_1\,, & \delta a_b &= \mathcal{D}_b\epsilon_2\,, \\ \delta\Phi &= -iq\epsilon_2\Phi\,, & \delta\Pi &= iq\epsilon_2\Pi\,, \\ \delta\Phi^* &= iq\epsilon_2\Phi^*\,, & \delta\Pi^* &= -iq\epsilon_2\Pi^*\,. \end{aligned} \tag{5.98}$$

We can also determine the multipliers from the equations of motion. By considering $[\phi, H_T]_P$ we see that $v_\phi = \dot{\phi}$. Likewise, we note that $[a_b, H_T]_P$ gives the expression of Eq. (5.87) provided $\partial_b v_2 = 0$. This allows us to set $v_2 = 0$ without any loss of generality.

5.3.2 Hamiltonian BRST derivation of the Path Integral

Following Sec. (5.1.2), we will now consider the Hamiltonian BRST formalism by extending the phase space to include additional ghosts and their momenta. In addition to the fields considered in the previous section, we introduce the ghost $(\mathcal{C})$ and antighost $(\bar{\mathcal{C}})$ and their conjugate momenta $\left(\mathcal{P}\,,\bar{\mathcal{P}}\right)$ which satisfy

$$\begin{aligned}\left[\bar{\mathcal{P}}(x),\bar{\mathcal{C}}(y)\right]_P &= -\delta(x,y) = \left[\mathcal{P}(x),\mathcal{C}(y)\right]_P \;,\\ \mathrm{gh}\left(\mathcal{C}\right) &= 1 = \mathrm{gh}\left(\bar{\mathcal{P}}\right)\;,\\ \mathrm{gh}\left(\mathcal{P}\right) &= -1 = \mathrm{gh}\left(\bar{\mathcal{C}}\right)\;,\end{aligned} \tag{5.99}$$

where all other brackets involving the ghosts vanish and where gh ( ) indicates the ghost number of the argument. Following Eq. (5.36), we define the generator of BRST transformations Q_{BRST} for scalar electrodynamics in this phase space as

$$Q_{\mathrm{BRST}} = \int\limits_\Sigma dV_x \left(\mathcal{C}(x)\Omega_2(x) - i\bar{\mathcal{P}}(x)\pi_\phi(x)\right)\;. \tag{5.100}$$

The BRST transformations of the fields follows from its Poisson bracket with Q_{BRST}, where as before the smearing functions are assumed to be regular at the horizon. We find

$$\begin{aligned} sa_b &= \mathcal{D}_b\mathcal{C}\,, & s\phi &= -i\bar{\mathcal{P}}\,,\\ s\bar{\mathcal{C}} &= i\pi_\phi\,, & s\mathcal{P} &= -\Omega_2\,,\\ s\Phi &= -iq\mathcal{C}\Phi\,, & s\Pi &= iq\mathcal{C}\Pi\,,\\ s\Phi^* &= iq\mathcal{C}\Phi^*\,, & s\Pi^* &= -iq\mathcal{C}\Pi^*\,,\\ && s\bar{\mathcal{P}} = 0 = s\mathcal{C}\,,\\ && s\pi^\phi = 0 = s\pi^a\,.\end{aligned} \tag{5.101}$$

As in the case of the Yang-Mills field in the previous section, we note that Eq. (5.101) involves no corrections from the horizons of the background. We now define a gauge fixing fermion Ψ as in Eq. (5.19)

$$\Psi = \int\limits_\Sigma dV_x \left(i\bar{\mathcal{C}}(x)\chi(x) + \mathcal{P}(x)\phi(x)\right)\;. \tag{5.102}$$

We will now make the following choice for the term χ involved in Eq. (5.102)

$$\chi = \mathcal{D}^a(\lambda a_a) + \frac{1}{2}\lambda\pi^\phi - n_a a^a\Big|_{\mathcal{H}}, \tag{5.103}$$

Apart from the surface term in Eq. (5.103), this corresponds to the Feynman gauge, which recovers the action for scalar electrodynamics with the Lorenz gauge fixing action. The additional surface term in Eq. (5.103) is allowed on backgrounds with horizons, whose implications we will now consider. By evaluating the Poisson bracket $[\Psi, Q_{\text{BRST}}]_P$ we find

$$[\Psi, Q_{\text{BRST}}]_P = -\left(\pi^\phi\chi + i\mathcal{P}\bar{\mathcal{P}} + \phi\Omega_2 + i\lambda\mathcal{D}_a\bar{\mathcal{C}}\mathcal{D}^a\mathcal{C} + i\bar{\mathcal{C}}n_a\mathcal{D}^a\mathcal{C}\Big|_{\mathcal{H}}\right). \tag{5.104}$$

Following Eq. (5.16), we can define the BRST invariant effective Hamiltonian

$$H_{eff} = H_0 - [\Psi, Q_{\text{BRST}}]_P\,, \tag{5.105}$$

which further allows us to construct the effective action from the Legendre transform

$$S_{eff} = \int dt \int_\Sigma dV_x \left(\dot{a}_b\pi^b + \lambda^{-1}\dot{\phi}\pi^\phi + \dot{\Phi}\Pi + \dot{\Phi}^*\Pi^* + \lambda^{-1}\dot{\bar{\mathcal{C}}}\bar{\mathcal{P}} + \dot{\mathcal{C}}\mathcal{P} - H_{eff}\right), \tag{5.106}$$

Let $\mu^\alpha \equiv \left(a_a, \pi^a, \phi, \pi^\phi, \Phi, \Pi, \Phi^*, \Pi^*, \bar{\mathcal{C}}, \bar{\mathcal{P}}, \mathcal{C}, \mathcal{P}\right)$ span over all the field variables. We can then define the path integral in the following way

$$\begin{aligned}
Z &= \int [\mathcal{D}\mu^\alpha] \exp\left(iS_{eff}\right) \\
&= \int [\mathcal{D}\mu^\alpha] \exp\Bigg(i\int dt \int_\Sigma dV_x \Bigg(\dot{a}_a\pi^a + \lambda^{-1}\dot{\phi}\pi^\phi + \dot{\Phi}\Pi + \dot{\Phi}^*\Pi^* + \lambda^{-1}\dot{\bar{\mathcal{C}}}\bar{\mathcal{P}} + \dot{\mathcal{C}}\mathcal{P} - \phi\Omega_2 \\
&\qquad - \lambda\left(\frac{1}{2}\pi^b\pi_b + \frac{1}{4}f_{ab}f^{ab} + \Pi\,\Pi^* + m^2\Phi\Phi^* + \bar{D}_a\Phi\left(\bar{D}^a\Phi\right)^* + i\mathcal{D}_a\bar{\mathcal{C}}\mathcal{D}^a\mathcal{C}\right) \\
&\qquad - \pi^\phi\chi - i\mathcal{P}\bar{\mathcal{P}}\Bigg) - i\int dt \oint_{\partial\Sigma} da_x\,\bar{\mathcal{C}}n_a\mathcal{D}^a\mathcal{C}\Bigg)\,,
\end{aligned} \tag{5.107}$$

where $[\mathcal{D}\mu^\alpha]$ is the path integral measure. By integrating out the momenta $\bar{\mathcal{P}}$, $\mathcal{P}$, Π, Π^* and π^a in Eq. (5.107), we find

$$Z = \int [\mathcal{D}\tilde{\mu}^\alpha] \exp\left(i \int dt \int\limits_\Sigma dV_x \left(\lambda \left(\frac{1}{2} e^a e_a - \frac{1}{4} f_{ab} f^{ab} + \lambda^{-2} D_0\Phi \left(D_0\Phi\right)^* - \bar{D}_a\Phi \left(\bar{D}^a\Phi\right)^* \right.\right.\right.$$
$$\left.\left. -m^2\Phi\Phi^* + i\lambda^{-2}\dot{\bar{\mathcal{C}}}\dot{\mathcal{C}} - i\mathcal{D}_a\bar{\mathcal{C}}\mathcal{D}^a\mathcal{C} \right) + \lambda\pi^\phi \left(\lambda^{-2}\dot{\phi} - \lambda^{-1}\mathcal{D}^a(\lambda a_a) - \frac{1}{2}\pi^\phi \right) \right)$$
$$\left. + i \int dt \oint\limits_{\partial\Sigma} da_x n^a \left(\pi^\phi a_a - i\bar{\mathcal{C}}\mathcal{D}_a\mathcal{C} \right) \right), \tag{5.108}$$

where $\tilde{\mu}^\alpha \equiv \left(a_a, \phi, \pi^\phi, \Phi, \Phi^*, \bar{\mathcal{C}}, \mathcal{C}\right)$ in Eq. (5.108). In deriving the expression in Eq. (5.108), we made use of Eq. (5.87) and the second line of Eq. (5.89) following the path integration of π^a, Π and Π^*. We can further express Eq. (5.108) in the following way

$$Z = \int \left[\mathcal{D}a_a\, \mathcal{D}\phi\, \mathcal{D}\Phi\, \mathcal{D}\Phi^*\, \mathcal{D}\bar{\mathcal{C}}\, \mathcal{D}\mathcal{C}\right] \exp\left(iS_{SED} + iS_{gh} + iS_{gf}\right), \tag{5.109}$$

where

$$\begin{aligned}
S_{SED} &= \int dt \int\limits_\Sigma dV_x \lambda \left(-h^{ab}\bar{D}_a\Phi\left(\bar{D}_b\Phi^*\right) + \lambda^{-2}D_0\Phi(D_0\Phi)^* - m^2\Phi\Phi^* - \frac{1}{4}f_{ab}f^{ab} + \frac{1}{2}e_a e^a \right), \\
S_{gf} &= \int dt \int\limits_\Sigma dV_x \lambda\pi^\phi \left(\lambda^{-2}\dot{\phi} - \lambda^{-1}\mathcal{D}^a(\lambda a_a) - \frac{1}{2}\pi_\phi \right) + \int dt \oint\limits_{\partial\Sigma} da_x n^a a_a \pi^\phi, \\
S_{gh} &= i\int dt \int\limits_\Sigma dV_x\, \lambda \left(\lambda^{-2}\dot{\bar{\mathcal{C}}}\dot{\mathcal{C}} - \mathcal{D}^a\bar{\mathcal{C}}\mathcal{D}_a\mathcal{C} \right) - i\int dt \oint\limits_{\partial\Sigma} da_x n_a \bar{\mathcal{C}}\mathcal{D}_a\mathcal{C}.
\end{aligned} \tag{5.110}$$

The BRST transformations in Eq. (5.101) now reduce to

$$\begin{aligned}
sa_a &= \mathcal{D}_a\mathcal{C}, \qquad s\phi = \dot{\mathcal{C}}, \\
s\Phi &= -iq\mathcal{C}\Phi, \qquad s\Phi^* = iq\mathcal{C}\Phi^*, \\
s\mathcal{C} &= 0, \qquad s\bar{\mathcal{C}} = i\pi^\phi.
\end{aligned} \tag{5.111}$$

5.3.3 Dressed charges from the co-BRST charge

Physical states $|\Phi\rangle$ in the BRST formalism satisfy $Q_{\rm BRST}|\Phi\rangle = 0$. However, due to the nilpotency of BRST transformations $s^2 = 0$, an equivalent class of states satisfies this condition with respect to the BRST charge. For instance, given any state $|P\rangle$ which satisfies $Q_{\rm BRST}|P\rangle = 0$, the state $|\widetilde{P}\rangle = Q_{\rm BRST}|T\rangle$ also satisfies $Q_{\rm BRST}|\widetilde{P}\rangle = 0$. If V denotes the inner product space of the theory, it can be subdivided into a direct sum of singlets and doublets, which are defined as follows

$$|P\rangle \in V \text{ is a singlet if } Q_{\rm BRST}|P\rangle = 0\,, |P\rangle \neq Q_{\rm BRST}|T\rangle = 0 \text{ for any } |T\rangle \in V$$

$$|\widetilde{P}\rangle\,, |T\rangle \in V \text{ is a doublet if } |\widetilde{P}\rangle = Q_{\rm BRST}|T\rangle \neq 0\,.$$

This decomposition can always be made unique and the physical (singlet) state space is a representation of the BRST cohomology Ker $(Q_{\rm BRST})$ /Im $(Q_{\rm BRST})$ [77].

The co-BRST charge can also be used to determine gauge invariant charges of the theory, which will mainly concern us in this subsection. One of the earliest proposals for a co-BRST charge in the context of the Lagrangian for quantum electrodynamics was provided in [76]. It was argued that a nilpotent operator $Q^{\perp}_{\rm BRST}$ (other than $Q_{\rm BRST}$) which reduces the ghost number by one and which preserves the gauge fixing action could be used to resolve singlet states. Physical states now need to satisfy $Q_{\rm BRST}|\Phi\rangle = 0$ and $Q^{\perp}_{\rm BRST}|\Phi\rangle = 0$. In the case of scalar electrodynamics in flat spacetime, the following dressed fields

$$\Phi_{phys} = \Phi \exp\left(iq\frac{\partial_i A^i}{\nabla^2}\right)\,, \qquad \Phi^*_{phys} = \Phi^* \exp\left(-iq\frac{\partial_i A^i}{\nabla^2}\right)\,, \tag{5.112}$$

satisfy the condition $Q_{\rm BRST}\,\Phi_{phys} = 0 = Q^{\perp}_{\rm BRST}\,\Phi_{phys}$ and $Q_{\rm BRST}\,\Phi^*_{phys} = 0 = Q^{\perp}_{\rm BRST}\,\Phi^*_{phys}$, while Φ and Φ^* do not. In Eq. (5.112), the index 'i' denotes spatial coordinates, while ∇^{-2} is the inverse Laplacian of flat spacetime which satisfies $\nabla^2_x\nabla^{-2}(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y})$, where $\delta(\vec{x}-\vec{y})$ is the Dirac delta function on flat spacetime.

The dressed fields given in Eq. (5.112) were first constructed by Dirac as gauge invariant variables of quantum electrodynamics (QED) in flat spacetime [108]. The application of dressed fields to resolve the infrared problem in QED was initiated in [68, 69] and were later shown to define asymptotic states and eliminate IR divergences in [70, 74]. More recently, the dressed fields have been used to provide finite QED scattering amplitudes on asymptotically flat spacetimes [73].

Within the Hamiltonian BRST formalism, the co-BRST operator can be identified with the gauge fixing fermion χ [77, 78]. We noted in the introduction of this chapter that regardless of the choice of Ψ, we can always derive a Hamiltonian and Lagrangian invariant under the BRST transformations generated by $Q_{\rm BRST}$. However it is possible to choose Ψ in such a way that it is also a conserved charge of the theory which generates its own nilpotent transformations. Unlike the BRST charge operator $Q_{\rm BRST}$ which follows from a prescribed generalization of the first class constraints [95], the co-BRST operator is not unique and there may exist several possible constructions.

Following Eq. (5.16), we now seek the following Hamiltonian

$$H_{eff} = \widetilde{H}_0 - \left[\widetilde{\Psi}, Q_{\rm BRST}\right]_P , \tag{5.113}$$

where $\widetilde{\Psi}$ is a co-BRST charge which generates its own nilpotent transformations and $\widetilde{H}_0$ is both BRST and co-BRST invariant. For scalar electrodynamics, we will assume that $Q_{\rm BRST}$ is as defined in Eq. (5.100). To simplify the notation in the following treatment let us define $J^0 = iq\,(\Phi^*\Pi^* - \Phi\Pi)$. Then the Gauss law constraint of scalar electrodynamics in Eq. (5.96) can be written as

$$\Omega_2 = n_b \pi^b\Big|_{\mathcal{H}} - \mathcal{D}_b \pi^b + J^0 . \tag{5.114}$$

The co-BRST operator, just as the dressed charges they help identify, will have a non-local contribution. We thus introduce a Green function $G(x, y)$ which satisifes

$$F(\mathcal{D})G(x, y) = \delta(x, y)\,, \tag{5.115}$$

where $F(\mathcal{D})$ is a differential operator which will be determined. We can now use the Green function to define $\widetilde{H}_0$ and $\widetilde{\Psi}$ as

$$\begin{aligned}\widetilde{H}_0 = H_0 + \frac{1}{2}\Bigg(&\int\limits_{\Sigma} dV_x \int\limits_{\Sigma} dV_y \left(\mathcal{D}^x_a \pi^a(x)\right) G(x, y) \left(\mathcal{D}^y_b \pi^b(y)\right) \\ &- \oint\limits_{\partial\Sigma} da_x \int\limits_{\Sigma} dV_y n^x_a \pi^a(x) G(x, y) \left(\mathcal{D}^y_b \pi^b(y)\right) \\ &- \int\limits_{\Sigma} dV_x \oint\limits_{\partial\Sigma} da_y \left(\mathcal{D}^x_a \pi^a(x)\right) G(x, y) n^y_b \pi^b(y) \\ &+ \oint\limits_{\partial\Sigma} da_x \oint\limits_{\partial\Sigma} da_y n^x_a \pi^a(x) G(x, y) n^y_b \pi^b(y)\Bigg)\end{aligned} \tag{5.116}$$

and

$$\begin{aligned}\widetilde{\Psi} &= \Psi + \frac{1}{2}\int\limits_{\Sigma} dV_x \mathcal{P}(x) \int\limits_{\Sigma} dV_y G(x, y)\Omega_2(y) \\ &= \Psi - \int\limits_{\Sigma} dV_x \mathcal{P}(x) \left(\frac{1}{2}\int\limits_{\Sigma} dV_y \left(\mathcal{D}^y_a \pi^a(y) - J^0(y)\right) G(x, y) - \oint\limits_{\partial\Sigma} da_y n^y_a \pi^a(y) G(x, y)\right).\end{aligned} \tag{5.117}$$

We will assume that H_0 and Ψ in the above equations are those given in Eq. (5.91) and Eq. (5.102), respectively. However, we will now consider a gauge fixing function χ which is different from that given in Eq. (5.103). We define

$$\chi = \mathcal{D}^a(\lambda^{-1} a_a) - \frac{1}{2}\pi^\phi - \lambda^{-1} n_a a^a\Big|_{\mathcal{H}}\,. \tag{5.118}$$

We make this choice as it provides the simplest construction of a co-BRST charge.

We can see that the terms in the parenthesis in Eq. (5.116) vanish under the BRST transformations given in Eq. (5.101), since $s\pi^a = 0$. Thus $\widetilde{H}_0$ is BRST invariant and Eq. (5.113) defines a BRST invariant effective action. We will now demonstrate that $\widetilde{\Psi}$ can generate its own nilpotent symmetry transformations under which $\widetilde{H}_0$ and hence H_{eff} are also invariant. By using Eq. (5.118) in Eq. (5.102), we can determine the transformations generated by $\widetilde{\Psi}$. Let us denote $\left[\mu^\alpha, \widetilde{\Psi}\right]_P = \delta^\perp \mu^\alpha$ as the transformations generated by $\widetilde{\Psi}$, where $\mu^\alpha \equiv \left(a_a, \pi^a, \phi, \pi^\phi, \Phi, \Pi, \Phi^*, \Pi^*, \bar{\mathcal{C}}, \bar{\mathcal{P}}, \mathcal{C}, \mathcal{P}\right)$ represents the set of all fields in the extended phase space. Evaluating the Poisson brackets of the fields with $\widetilde{\Psi}$ we find the following set of transformations

$$
\begin{aligned}
\delta^\perp a_b(x) &= \int\limits_\Sigma dV_y \frac{1}{2}\mathcal{D}^x_b(G(x,y))\mathcal{P}(y)\,, &\qquad \delta^\perp\phi(x) &= -\frac{1}{2}i\bar{\mathcal{C}}(x)\,,\\
\delta^\perp\mathcal{C}(x) &= -\phi(x) - \int\limits_\Sigma dV_y \frac{1}{2}\Omega_2(y)G(x,y), & \delta^\perp\bar{\mathcal{P}}(x) &= -i\chi(x)\,,\\
\delta^\perp\pi_a(x) &= i\lambda^{-1}(x)D^x_a\bar{\mathcal{C}}(x)\,, & \delta^\perp\pi_\phi(x) &= -\mathcal{P}(x)\,,\\
\delta^\perp\Phi(x) &= -\int\limits_\Sigma dV_y \frac{1}{2}iq\mathcal{P}(y)\Phi(x)G(x,y)\,, & \delta^\perp\Pi(x) &= \int\limits_\Sigma dV_y \frac{1}{2}iq\mathcal{P}(y)\Pi(x)G(x,y)\,,\\
\delta^\perp\Phi^*(x) &= \int\limits_\Sigma dV_y \frac{1}{2}iq\mathcal{P}(y)\Phi^*(x)G(x,y)\,, & \delta^\perp\Pi^*(x) &= -\int\limits_\Sigma dV_y \frac{1}{2}iq\mathcal{P}(y)\Pi^*(x)G(x,y)\,,
\end{aligned}
$$
$$
\delta^\perp\bar{\mathcal{C}}(x) = 0 = \delta^\perp\mathcal{P}(x)\,. \tag{5.119}
$$

The Poisson bracket with $\widetilde{\Psi}$ reduces the ghost number of the field it acts on by 1. These transformations are nilpotent for all the fields μ^α, i.e. $(\delta^\perp)^2\mu^\alpha = 0$, provided the Green function $G(x,y)$ satisfies

$$
\int\limits_\Sigma dV_x f(y)\mathcal{D}^x_a\left(\lambda^{-1}(x)\mathcal{D}^a_x G(x,y)\right) - \oint\limits_{\partial\Sigma} da_x\, f(y) n^a_x \lambda^{-1}(x)\mathcal{D}^x_a G(x,y) = f(x)\,, \tag{5.120}
$$

where $f(x)$ is any well behaved function on the hypersurface Σ. We can equivalently write Eq. (5.120) in the following way

$$\mathcal{D}^x_a \left(\lambda^{-1}(x)\mathcal{D}^a_x G(x,y)\right) - \lambda^{-1} n^a_x \mathcal{D}^x_a G(x,y) = \delta(x,y)\,. \tag{5.121}$$

Thus $F(\mathcal{D}) = \mathcal{D}_a\lambda^{-1}\mathcal{D}^a - n^a\lambda^{-1}\mathcal{D}_a\Big|_{\mathcal{H}}$ in Eq. (5.115). It is straightforward to verify that $\delta^\perp\chi = 0$, $\delta^\perp\widetilde{H}_0 = 0$ and $\delta^\perp H_{eff} = 0$. Since the transformations generated by $\widetilde{\Psi}$ are nilpotent and preserves the BRST invariant Hamiltonian, $\widetilde{\Psi}$ satisfies all the properties of a co-BRST operator. The dressed charges now take the form

$$\begin{aligned}
\Phi_{phys}(x) &= \Phi(x)\exp\left(iq\int\limits_{\Sigma} dV_z\,\mathcal{D}^z_a\left(\lambda^{-1}(z)a^a(z)\right)G(x,z) - iq\oint\limits_{\partial\Sigma} da_z\,\lambda^{-1}(z)n^z_a a^a(z)G(x,z)\right)\,,\\
\Phi^*_{phys}(x) &= \Phi^*(x)\exp\left(-iq\int\limits_{\Sigma} dV_z\,\mathcal{D}^z_a\left(\lambda^{-1}(z)a^a(z)\right)G(x,z) + iq\oint\limits_{\partial\Sigma} da_z\,\lambda^{-1}(z)n^z_a a^a(z)G(x,z)\right)\,.
\end{aligned} \tag{5.122}$$

Using Eq. (5.101) and Eq. (5.119) on Eq. (5.122), we note that $s\Phi_{phys} = 0 = \delta^\perp\Phi_{phys}$ and likewise for $s\Phi^*_{phys} = 0 = \delta^\perp\Phi^*_{phys}$. The surface integrals in Eq. (5.122) now account for contributions from the horizons of the spacetime. In particular, the above expressions for dressed fields generalize Eq. (5.112) to backgrounds with a cosmological horizon, such as Schwarzschild-de Sitter. Thus the co-BRST charge can be used to identify dressed matter fields on non-asymptotically flat backgrounds.

We could have constructed another co-BRST charge using χ as given in Eq. (5.103). In any choice of χ other than Eq. (5.118), this will require a careful inclusion of factors of λ in the definition of $\widetilde{H}_0$ and $\widetilde{\Psi}$. The differential operator $F(\mathcal{D})$ in Eq. (5.115) would also be modified and additional boundary conditions on $G(x,y)$ would be needed at the horizons.

5.4 Covariant effective actions

The effective actions derived using the Hamiltonian BRST approach in the previous sections can be expressed as spacetime covariant integrals over $\mathcal{M}$. In the case of the Yang-Mills field, we derived the effective action comprising $S_{YM}\,, S_{gf}$ and S_{gh} in Eq. (5.46), Eq. (5.47) and Eq. (5.48) respectively. Using Eq. (5.1) in Eq. (5.46), we find the usual spacetime covariant action of the Yang-Mills field

$$S_{YM} = \int dt \int\limits_{\Sigma} dV_x \, \lambda \left(\frac{1}{2} e^a_A e^A_a - \frac{1}{4} f^A_{ab} f^{ab}_A \right) = - \int\limits_{\mathcal{M}} dV_4 \, \frac{1}{4} F^A_{ab} F^{ab}_A \tag{5.123}$$

Using Eq. (5.1) and Eq. (5.6) in Eq. (5.47) and Eq. (5.48), we find

$$\begin{aligned}
S_{gh} = i \int dt \int\limits_{\Sigma} dV_x \, \lambda & \left(\lambda^{-2} \dot{\bar{\mathcal{C}}}_A \left(\dot{\mathcal{C}}^A + g C^A_{BC} \mathcal{C}^B \phi^C \right) - \mathcal{D}^a \bar{\mathcal{C}}_A (\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a) \right) \\
& - i \int dt \oint\limits_{\partial\Sigma} da_x \, n_a \bar{\mathcal{C}}_A \left(\mathcal{D}_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B a^C_a \right) \\
= -i \int\limits_{\mathcal{M}} dV_4 \, \nabla^a \bar{\mathcal{C}}_A & \left(\nabla_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B A^C_a \right) - i \oint\limits_{\mathcal{H}} da_3 \, \frac{1}{\sqrt{2}} \left(l^a - k^a \right) \bar{\mathcal{C}}_A \left(\nabla_a \mathcal{C}^A + g C^A_{BC} \mathcal{C}^B A^C_a \right) \, ; \\
S_{gf} = \int dt \int\limits_{\Sigma} dV_x \, \lambda \pi^\phi_A & \left(\lambda^{-2} \dot{\phi}^A - \lambda^{-1} \mathcal{D}^a (\lambda a^A_a) - \frac{1}{2} \pi^A_\phi \right) + \int dt \oint\limits_{\partial\Sigma} da_x \, n^a a^A_a \pi^\phi_A \\
= - \int\limits_{\mathcal{M}} dV_4 \, \pi^\phi_A & \left(\pi^\phi_A + \nabla^a A^A_a \right) + \oint\limits_{\mathcal{H}} da_3 \, \frac{1}{\sqrt{2}} \left(l^a - k^a \right) A^A_a \pi^\phi_A \, .
\end{aligned} \tag{5.124}$$

In Eq. (5.124), $\oint\limits_{\mathcal{H}}$ refers to surface integrals over the horizons of the spacetime and $da_3 = dt da_x$.

We can similarly determine the spacetime covariant effective actions of scalar electrodynamics. By using Eq. (5.1) and Eq. (5.6) in Eq. (5.110) we find

$$\begin{aligned} S_{SED} &= \int dt \int_\Sigma dV_x \, \lambda \left(-h^{ab} \bar{D}_a \Phi \left(\bar{D}_b \Phi^* \right) + \lambda^{-2} D_0 \Phi (D_0 \Phi)^* - m^2 \Phi \Phi^* - \frac{1}{4} f_{ab} f^{ab} + \frac{1}{2} e_a e^a \right) \\ &= - \int_{\mathcal{M}} dV_4 \, \left(D_a \Phi (D_b \Phi)^* g^{ab} + m^2 \Phi \Phi^* + \tfrac{1}{4} F_{ab} F_{cd} g^{ac} g^{bd} \right) \, , \end{aligned} \tag{5.125}$$

and

$$\begin{aligned} S_{gf} &= \int dt \int_\Sigma dV_x \, \lambda \pi^\phi \left(\lambda^{-2} \dot{\phi} - \lambda^{-1} \mathcal{D}^a (\lambda a_a) - \frac{1}{2} \pi_\phi \right) + \int dt \oint_{\partial\Sigma} da_x \, n^a a_a \pi^\phi \\ &= - \int_{\mathcal{M}} dV_4 \, \pi^\phi \left(\pi^\phi + \nabla^a A_a \right) + \oint_{\mathcal{H}} da_3 \, \frac{1}{\sqrt{2}} \left(l^a - k^a \right) A_a \pi^\phi \, , \\ S_{gh} &= i \int dt \int_\Sigma dV_x \, \lambda \left(\lambda^{-2} \dot{\bar{\mathcal{C}}} \dot{\mathcal{C}} - \mathcal{D}^a \bar{\mathcal{C}} \mathcal{D}_a \mathcal{C} \right) - i \int dt \oint_{\partial\Sigma} da_x \, n_a \bar{\mathcal{C}} \mathcal{D}_a \mathcal{C} \\ &= -i \int_{\mathcal{M}} dV_4 \, \nabla^a \bar{\mathcal{C}} \nabla_a \mathcal{C} - i \oint_{\mathcal{H}} da_3 \, \frac{1}{\sqrt{2}} \left(l^a - k^a \right) \bar{\mathcal{C}} \nabla_a \mathcal{C} \, . \end{aligned} \tag{5.126}$$

The actions considered in Eq. (5.123), Eq. (5.124), Eq. (5.125) and Eq. (5.126) could be useful within the Lagrangian BRST formalism. They also make explicit the fact that for a given gauge theory on a curved background, such as the Yang-Mills field or scalar electrodynamics, the ghost and gauge fixing covariant actions will involve surface integrals over the horizons of the background. Such surface integrals do not arise when the curved background only involves spatial boundaries.

5.5 Discussion

In this chapter, we considered the Hamiltonian BRST formalism for constrained field theories on spherically symmetric backgrounds with one or more horizons. By considering the Yang-Mills field and scalar electrodynamics as examples, we first applied the Dirac-Bergmann formalism to derive the Gauss law constraint which involves contributions from the horizons of the background. We then extended the phase space to include the ghosts and their multipliers, needed in order to carry out the Hamiltonian BRST formalism. The BRST charge is an extension of the first class constraints of the theory and involves the horizon contribution present in the Gauss law constraint. In both examples, this led us to consider gauge fixing functions which include contributions from the horizons. The resulting effective action of the theory, as a consequence, now involves surface integrals over the horizons in the ghost and gauge fixing actions.

In the case of the Yang-Mills field, we proceeded to consider the effect of these surface integrals on the renormalizability of the theory. The Zinn-Justin equation was used to show that the quantum BRST transformations, which leaves the path integral in the presence of sources invariant, have the same form as those which leave the tree level action of the theory invariant. We then considered a quantum effective action which include bulk and horizon contributions, whose terms were determined based on symmetries and power counting arguments. We determined that the surface integrals in the ghost and gauge fixing effective actions, derived in Eq. (5.47) and Eq. (5.48), do not affect the renormalizability of the theory. Furthermore, we were able to conclude that the bulk and surface actions are in general separately renormalizable, with the gauge and ghost fields allowed to have an arbitrary behaviour at the horizon. We can also note from the quantum effective action in Eq. (5.82) and its variation under quantum BRST transformations in Eq. (5.83) that we could have involved the lapse function λ in such a way that the gauge fixing action has no surface integral while the ghost action does, and vice versa. These would constitute special cases that are a consequence of the vanishing lapse function on the horizon. Our results for the renormalizability of the Yang-Mills field should be contrasted with results on manifolds with spatial boundaries. On these backgrounds, the BRST charge operator is unaffected by the presence of boundaries and as a consequence, fields must be made to satify BRST invariant boundary conditions to ensure the invariance of the path integral under BRST transformations [109–111].

We also considered the description of dressed charges in the case of scalar electrodynamics on spherically symmetric backgrounds with horizons. This was determined through the co-BRST charge of the theory, constructed from the gauge fixing fermion $\widetilde{\Psi}$ in the Hamiltonian BRST formalism. When the gauge fixing term χ in $\widetilde{\Psi}$ involves additional surface contributions due to the horizons of the spacetime, as in Eq. (5.117), we determined that the dressing of the fields is also modified by corresponding terms at the horizons of the background as in Eq. (5.122).

Our result concerning dressed charges motivates further investigations into the nature of the electric field and physical propagators of the theory near the horizon. Such calculations have been well understood in flat spacetime and in addition, play an important role in determining if the dressings are physically viable [74]. For instance, the following dressed field in flat spacetime can appear perfectly legitimate

$$\Phi_{phys}(x) = \Phi(x)\text{exp}\left(\int_{\Gamma}^{x} dz^i A_i(x_0, z)\right) \ ,$$

where the integral in the exponent is over some path Γ. However, this dressing provides an infinitely excited state, where the electric flux is confined along Γ. On the other hand, the field given in Eq. (5.112) does provide the correct expression for the electric field of a static charge. With an appropriate dressing identified, the propagators for dressed fields can be used to investigate infrared properties and soft limits [75]. In the context of the dressed fields in Eq. (5.122), the electric field must be such that it vanishes for an observer on the horizon, consistent with the Gauss law constraint. These dressed fields will also have an interesting infrared behaviour and soft limit near the horizons. As noted in the introduction of this thesis, the dressed charges in Eq. (5.112) do provide a realization of the soft charges at null infinity and have soft limits consistent with Weinberg's soft photon theorem [73]. Similar soft charges have been argued to exist on the horizons of black holes in [67], but whose construction have not yet been realized.

6 Summary and future outlook

In this thesis, we considered the constrained dynamics of field theories on curved backgrounds with horizons. Our Hamiltonian analysis was carried out on foliated spacetimes, where spatial sections of Killing horizons constituted boundaries of the spatial hypersurfaces. However, these backgrounds differ from manifolds with spatial boundaries. While gauge parameters can be fixed due to the regularity of fields at the spatial boundaries of manifolds, such conditions need not be required of fields on Killing horizons. More specifically, horizons constitute a boundary to the *observations* of static or stationary observers of the background and not a physical boundary of the manifold. We can only require that gauge invariant scalars are finite on the horizons and need not vanish. Gauge fields on the other hand, some of which appear as the multipliers of the constraints in the Hamiltonian, can have an arbitrary behaviour at the horizon. This property was reflected in our derivation of the constraints using the Dirac-Bergmann formalism, wherein we made no particular assumptions on the smearing functions at the horizon except that they were well behaved. This results in the derivation of constraints which involve additional contributions from the horizons of the background. This further affected the charges, dynamics and quantization of gauge theories on curved backgrounds with horizons.

In Chapter 2, we reviewed the Hamiltonian formulation of field theories on foliated backgrounds and provided the definitions for a manifestly covariant formalism on the spatial hypersurfaces. This chapter also reviewed essential elements of the Dirac-Bergmann formalism for constrained field theories used in this thesis. In Chapter 3, spherically symmetric backgrounds with horizons were considered, where the spatial hypersurfaces are orthogonal to the timelike Killing vector field of the background. As specific examples, we applied the Dirac-Bergmann formalism to the Maxwell field and Abelian Higgs model. In Chapter 4, we extended the Dirac-Bergmann formalism to a certain class of axially symmetric backgrounds which admit spatial hypersurfaces. The spatial hypersurfaces of these backgrounds are orthogonal to a timelike vector field which is a Killing vector field only on the horizons of the spacetime and nowhere else. Apart from this subtlety, the Dirac-Bergmann formalism was considered exactly as in the case of spherically symmetric backgrounds. In all theories considered on both spherically and axially symmetric backgrounds, we derived a Gauss law constraint which involve additional surface contributions from the Killing horizons of the background.

We explored specific implications of the horizon corrections to the Gauss law constraint on the charges, gauge fixing and Dirac brackets of the theory. Ordinarily in the case of charged black holes, one expects that the electric flux does not vanish outside and on the event horizon of the black hole. The surface terms at the horizon in the Gauss law results in an electric flux which is non-vanishing across any surface outside the horizon of the black hole, but which vanishes exactly across the event horizon. We argued that a possible explanation of this observation are the presence of equal and opposite charges on either side of the horizon. This would imply a screening effect exactly at the horizon which does not, however, affect the electric flux outside the horizon. The precise implications of the constraint we derived on the quantum state and the properties of fields at the horizons is a topic for future investigation.

The horizon contributions in the Gauss law constraint can also modify known gauge fixing choices. This was noted in the case of the radiation gauge for the Maxwell field about the Schwarzschild background in Chapter 3. We first showed that by not including horizon corrections in the radiation gauge, we can derive Dirac brackets of the theory which are the covariant generalization of those known in flat sapcetime. However, by explicitly considering the Green function of the spacetime Laplacian operator on the Schwarzschild background, we showed that this bracket reduces to the Poisson bracket when any one of the fields in the bracket is evaluated at the horizon. We thus also considered a modified radiation gauge which involves surface contributions from the horizons, analogous to those present in the Gauss law constraint. This resulted in Dirac brackets which remain distinct from the Poisson brackets even at the horizon of the Schwarzschild background. The Green function used to prove this result was derived in the appendix of Chapter 3.

It should be noted however that not all gauge fixing choices must always require horizon corrections in order to be distinct from Poisson brackets at the horizons. As shown in the case of the unitary gauge for the Abelian Higgs model in Chapter 3 and the axial gauge for the Maxwell field in Chapter 4, the Dirac brackets in these gauges remain distinct at all points of the spacetime, including the horizons. In these gauges, the dependent variables following the gauge fixing do involve corrections at the horizon due to the modified Gauss law constraint. This involves the fields π_η in Eq. (3.97) and ϕ in Eq. (4.75).

To further explore the effect of horizons on the quantization of gauge theories, we considered the Hamiltonian BRST formalism in Chapter 5. We addressed field theories on spherically symmetric backgrounds with horizons, whose first-class constraints satisfy a Lie algebra. As examples, we considered the Yang-Mills field and scalar electrodynamics. An effective BRST invariant action was derived through a choice of a gauge fixing fermion Ψ which involves the gauge fixing term χ. When χ involved surface contributions from the horizons of the background, we derived an effective action which involve surface integrals in the ghost and gauge fixing actions. We used the Zinn-Justin equation to show that quantum BRST transformations of all theories of the Yang-Mills type have the same form as the classical BRST transformations. By further considering a general form of the quantum effective action, we showed that the bulk and surface actions are separately renormalizable.

We also considered how the gauge fixing fermion can be chosen to generate its own nilpotent symmetry transformations, which preserves the gauge fixing term χ and the BRST invariant Hamiltonian. In this case, $\widetilde{\Psi}$ provides a representation of a co-BRST operator. For abelian gauge theories in general, the BRST and co-BRST operators can be used to identify physical states of the theory and gauge invariant fields. In the case of scalar electrodynamics in flat spacetime, scalar fields dressed with the exponential function of the Coulomb gauge term are separately invariant under BRST and co-BRST transformations. For curved backgrounds with horizons, the dressing function will involve additional surface contributions from the horizons due to horizon corrections present in the gauge fixing term. As such, the dressed charges derived in this thesis provide a generalization of the known dressed charges of quantum electrodynamics in flat spacetime.

As discussed at the end of the last chapter, the propagators of dressed fields will allow for an investigation of the soft limits of photons near the horizons of black holes. However, since the dressing is also non-local, correlation functions involving dressed fields at and outside the horizon could provide observable signatures for fields at horizons. The investigation of propagators and correlation functions resulting from a BRST and co-BRST invariant path integral might thus provide key insights into the nature of gauge invariant fields at the horizons of the background. While surface integrals in the gauge fixing action do modify dressed charges and could affect physical processes at the horizons, surface integrals in the ghost action can likewise have interesting consequences. We note that ghost fields in BRST invariant actions help define the partition function of thermal gauge theories, especially within real time approaches of thermal field theories such as the thermo-field double (TFD) formalism [112]. Thus surface integrals of the ghosts at the horizon could modify the known thermal propagators and correlation functions in flat spacetime and provide corrections near the horizons on black hole backgrounds.

We can also consider gravitational fields perturbatively about fixed backgrounds with horizons. Unlike gauge theories which were considered in this thesis, gravitational field perturbations must preserve the foliation and the horizons of the background. A particularly interesting avenue for future investigation involve perturbations about Kerr backgrounds, which will be relevant for gravitational waves. Dressed charges in gravity can also be similarly constructed as in abelian gauge theories [113]. Local field theories do not commute with the generators of diffeomorphisms and as such, cannot represent diffeomorphism-invariant observables in an effective theory of gravity [114]. Non-locally dressed fields on the other hand can be diffeomorphism-invariant [115]. It will be interesting to explore these constructions on backgrounds with horizons and their relation to soft graviton theorems.

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